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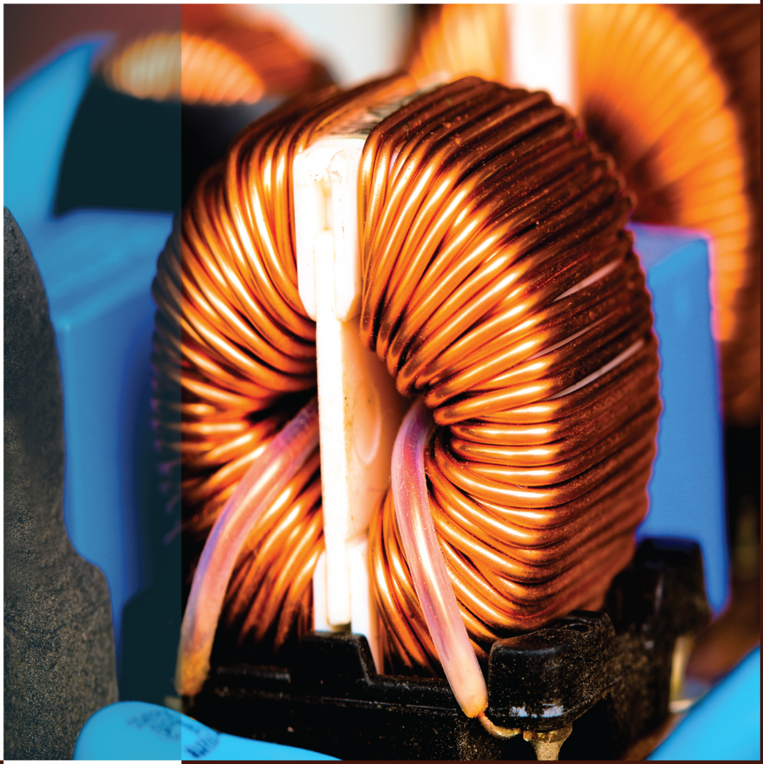

Engineering Electromagnetics

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CHAPTER 1

VECTOR ANALYSIS

INTRODUCTION

Vector analysis, a branch of mathematics that deals with quantities that have both magnitude and direction. Some physical and geometric quantities, called scalars, can be fully defined by specifying their magnitude in suitable units of measure. Thus, mass can be expressed in grams, temperature in degrees on some scale, and time in seconds. Scalars can be represented graphically by points on some numerical scale such as a clock or thermometer. There also are quantities, called vectors, that require the specification of direction as well as magnitude. Velocity, force, and displacement are examples of vectors. A vector quantity can be represented graphically by a directed line segment, symbolized by an arrow pointing in the direction of the vector quantity, with the length of the segment representing the magnitude of the vector.

1.1 DEFINITIONS OF VECTOR ANALYSIS

In science and engineering we frequently encounter quantities that have magnitude and magnitude only: mass, time, and temperature. These we label scalar quantities, which remain the same no matter what coordinates we use. In contrast, many interesting physical quantities have magnitude and, in addition, an associated direction. This second group includes displacement, velocity, acceleration, force, momentum, and angular momentum. Quantities with magnitude and direction are labeled vector quantities. Usually, in elementary treatments, a vector is defined as a quantity having magnitude and direction. To distinguish vectors from scalars, we identify vector quantities with boldface type, that is, V.

Our vector may be conveniently represented by an arrow, with length proportional to the magnitude. The direction of the arrow gives the direction of the vector, the positive sense of direction being indicated by the point. In this representation, vector addition

$$C = A + B \qquad (1)$$

consists in placing the rear end of vector B at the point of vector A. Vector C is then represented by an arrow drawn from the rear of A to the point of B. This procedure, the triangle law of addition, assigns meaning to Eq. (1) and is illustrated in Figure 1. By completing the parallelogram, we see that

$$C = A + B = B + A, \qquad (2)$$

as shown in Figure 2. In words, vector addition is commutative.

For the sum of three vectors

$D = A + B + C,$

Figure 3, we may first add A and B:

$A + B = E.$

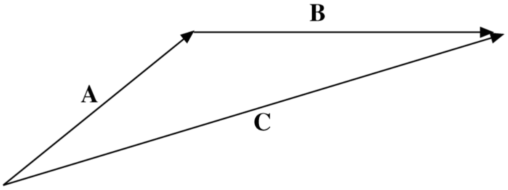


Figure 1. Triangle law of vector addition.

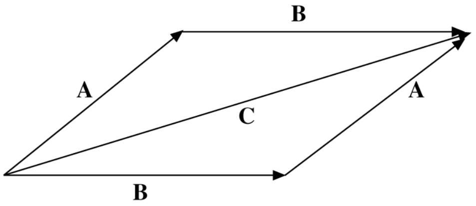


Figure 2. Parallelogram law of vector addition.

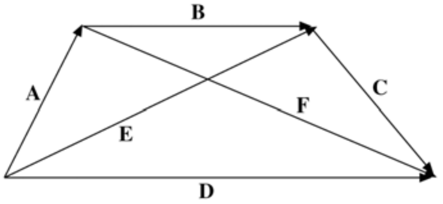


Figure 3. Vector addition is associative.

Then this sum is added to C:

D = E + C.

Similarly, we may first add B and C:

B + C = F.

Then

D = A + F.

In terms of the original expression,

(A + B) + C = A + (B + C).

Vector addition is associative.

A direct physical example of the parallelogram addition law is provided by a weight suspended by two cords. If the junction point (O in Figure 4) is in equilibrium, the vector

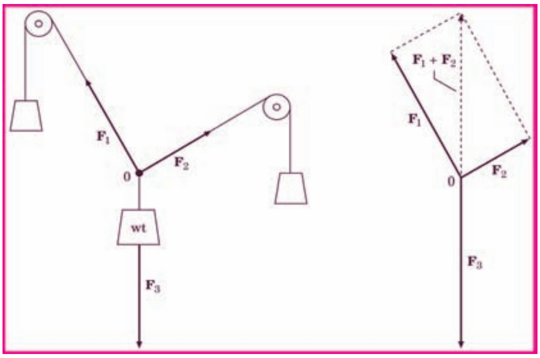


Figure 4. Equilibrium of forces: $F_1 + F_2 = -F_3$.

sum of the two forces F_1 and F_2 must just cancel the downward force of gravity, F_3. Here the parallelogram addition law is subject to immediate experimental verification.

Subtraction may be handled by defining the negative of a vector as a vector of the same magnitude but with reversed direction. Then

A − B = A + (−B).

In Figure 3,

A = E − B.

Note that the vectors are treated as geometrical objects that are independent of any coordinate system.

The representation of vector A by an arrow suggests a second possibility. Arrow A (Figure 5), starting from the origin, terminates at the point (A_x ,A_y ,A_z). Thus, if we agree that the vector is to start at the origin, the positive end may be specified by giving the Cartesian coordinates (A_x, A_y , A_z) of the arrowhead.

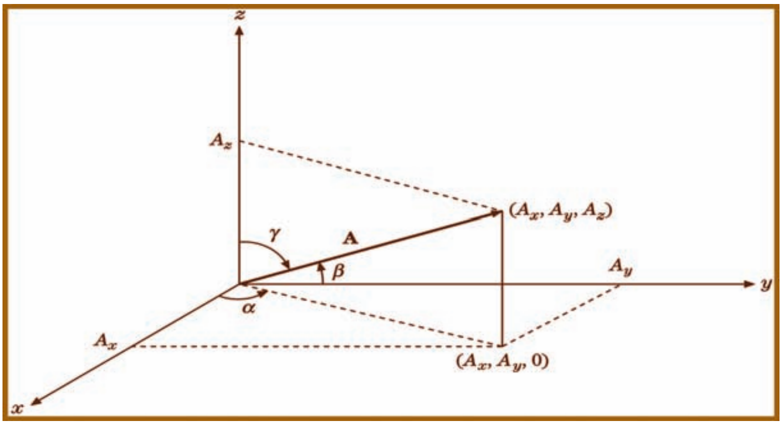


Figure 5. Cartesian components and direction cosines of A.

Although A could have represented any vector quantity (momentum, electric field, etc.), one particularly important vector quantity, the displacement from the origin to the point (x,y,z), is denoted by the special symbol r. We then have a choice of referring to the displacement as either the vector r or the collection (x,y,z), the coordinates of its endpoint:

$$r \leftrightarrow (x,y,z). \qquad (3)$$

Using r for the magnitude of vector r, we find that Figure 5 shows that the endpoint coordinates and the magnitude are related by

$$x = r \cos\alpha,\ y = r \cos\beta,\ z = r \cos \gamma. \qquad (4)$$

Here $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the direction cosines, α being the angle between the given vector and the positive x-axis, and so on. One further bit of vocabulary: The quantities A_x, A_y, and A_z are known as the (Cartesian) components of A or the projections of A, with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Thus, any vector A may be resolved into its components (or projected onto the coordinate axes) to yield $A_x = A\cos\alpha$, etc., as in Eq. (4). We may choose to refer to the vector as a single quantity A or to its components (A_x, A_y, A_z). Note that the subscript x in Ax denotes the x component and not a dependence on the variable x. The choice between using A or its components (A_x, A_y, A_z) is essentially a choice between a geometric and an algebraic representation. Use either representation at your convenience. The geometric "arrow in space" may aid in visualization. The algebraic set of components is usually more suitable for precise numerical or algebraic calculations.

Vectors enter physics in two distinct forms. (1) Vector A may represent a single force acting at a single point. The force of gravity acting at the center of gravity illustrates this form. (2) Vector A may be defined over some extended region; that is, A and its components may be functions of position: $A_x = A_x(x, y, z)$, and so on. Examples of this sort include the velocity of a fluid varying from point to point over a given volume and electric and magnetic fields. These two cases may be distinguished by referring to the vector defined over a region as a vector field. The concept of the vector defined over a region and being a function of position will become extremely important when we differentiate and integrate vectors.

At this stage it is convenient to introduce unit vectors along each of the coordinate axes. Let $\hat{x}$ be a vector of unit magnitude pointing in the positive x-direction, $\hat{y}$, a vector of unit magnitude in the positive y-direction, and $\hat{z}$ a vector of unit magnitude in the positive zdirection. Then $\hat{x}A_x$ is a vector with magnitude equal to $|A_x|$ and in the x-direction. By vector addition,

Note that if A vanishes, all of its components must vanish individually; that is, if

$$A = 0, \text{ then } A_x = A_y = A_z = 0. \qquad (5)$$

This means that these unit vectors serve as a basis, or complete set of vectors, in the threedimensional Euclidean space in terms of which any vector can be expanded. Thus, Eq. (5) is an assertion that the three unit vectors $\hat{x}, \hat{y}, \text{and } \hat{z}$ span our real three-dimensional space: Any vector may be written as a linear combination of $\hat{x}, \hat{y}, \text{and } \hat{z}$. Since $\hat{x}, \hat{y}, \text{and } \hat{z}$ are linearly independent (no one is a linear combination of the other two), they form a basis for the real three-dimensional Euclidean space. Finally, by the Pythagorean theorem, the magnitude of vector A is

$$|\mathbf{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}. \qquad (6)$$

Note that the coordinate unit vectors are not the only complete set, or basis. This resolution of a vector into its components can be carried out in a variety of coordinate systems. Here we restrict ourselves to Cartesian coordinates, where the unit vectors have the coordinates $\hat{x} = (1, 0, 0)$, $\hat{y} = (0, 1, 0)$ and $\hat{z} = (0, 0, 1)$ and are all constant in length and direction, properties characteristic of Cartesian coordinates.

As a replacement of the graphical technique, addition and subtraction of vectors may now be carried out in terms of their components. $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ and $\mathbf{B} = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z$,

$$\mathbf{A} \pm \mathbf{B} = \hat{\mathbf{x}}(A_x \pm B_x) + \hat{\mathbf{y}}(A_y \pm B_y) + \hat{\mathbf{z}}(A_z \pm B_z).$$

It should be emphasized here that the unit vectors $\hat{x}, \hat{y}, \text{and } \hat{z}$ are used for convenience. They are not essential; we can describe vectors and use them entirely in terms of their components: $A \leftrightarrow (A_x, A_y, A_z)$. However, $\hat{x}, \hat{y}, \text{and } \hat{z}$ emphasize the direction.

1.2 VECTOR ALGEBRA

The operations of addition, subtraction and multiplication familiar in the algebra of numbers (or scalars) can be extended to an algebra of vectors.

The following definitions and properties fundamentally define the vector:

1.2.1 Sum of Vectors

The addition of vectors a and b is a vector c formed by placing the initial point of b on the terminal point of a and then joining the initial point of a to the terminal point of b. The sum is written c = a + b . This definition is called the parallelogram law for vector addition because, in a geometrical interpretation of vector addition, c is the diagonal of a parallelogram formed by the two vectors a and b, Figure 6. The following properties hold for vector addition:

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ … commutative law

$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ … associative law

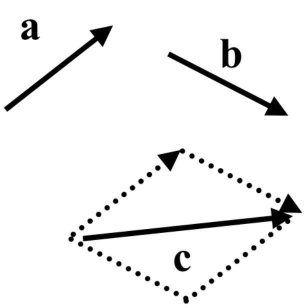


Figure 6: addition of vectors.

1.2.2 The Negative Vector

For each vector a there exists a negative vector. This vector has direction opposite to that of vector a but has the same magnitude; it is denoted by − a . A geometrical interpretation of the negative vector is shown in Figure 7a.

(a)

(b)

Figure 7: (a) negative of a vector; (b) subtraction of vectors.

1.2.3 Subtraction of Vectors and the Zero Vector

The subtraction of two vectors a and b is defined by a − b = a + (−b) , Figure 7b. If a = b then a − b is defined as the zero vector (or null vector) and is represented by the symbol o. It has zero magnitude and unspecified direction. A proper vector is any vector other than the null vector. Thus the following properties hold:

$$\mathbf{a}+\mathbf{o}=\mathbf{a}$$
$$\mathbf{a}+(-\mathbf{a})=\mathbf{o}$$

1.2.4 Scalar Multiplication

The product of a vector a by a scalar α is a vector αa with magnitude $|\alpha|$ times the magnitude of a and with direction the same as or opposite to that of a, according as α is positive or negative. If $\alpha = 0$, αa is the null vector. The following properties hold for scalar multiplication:

$(\alpha+\beta)\mathbf{a}=\alpha\mathbf{a}+\beta\mathbf{a}$ … distributive law, over addition of scalars
$\alpha(\mathbf{a}+\mathbf{b})=\alpha\mathbf{a}+\alpha\mathbf{b}$ … distributive law, over addition of vectors
$\alpha(\beta\mathbf{a})=(\alpha\beta)\mathbf{a}$ … associative law for scalar multiplication

Note that when two vectors a and b are equal, they have the same direction and magnitude, regardless of the position of their initial points. Thus a b = in Figure 8. A particular position in space is not assigned here to a vector – it just has a magnitude and a direction. Such vectors are called free, to distinguish them from certain special vectors to which a particular position in space is actually assigned.

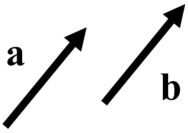


Figure 8: equal vectors.

The vector as something with "magnitude and direction" and defined by the above rules is an element of one case of the mathematical structure, the vector space.

1.3 COMPONENTS OF A VECTOR

Various components of a vector and the addition, multiplication comparison of vectors using components.

1.3.1 Unit Vector

Let's take a point each on the x, y, and z-axis as follows:

- A (1, 0, 0) on x-axis

- B (0, 1, 0) on y-axis and
- C (0, 0, 1) on z-axis

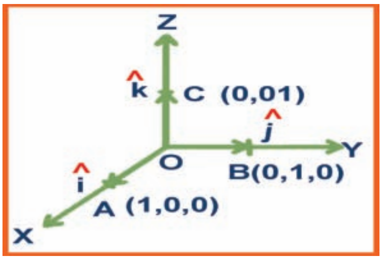


Figure 9.

So, we have

$$|\overrightarrow{OA}| = 1, |\overrightarrow{OB}| = 1, \text{ and } |\overrightarrow{OC}| = 1$$

These vectors $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OC}$, each having magnitude 1 are Unit Vectors along the axes OX, OY, and OZ respectively. They are denoted by $\vec{i}, \vec{j}$, and $\vec{k}$ as shown in Figure 9 above.

1.3.2 Component Form of a Vector

Let's consider a position vector $\overrightarrow{OP}$ of a point P (x, y, z) as shown below

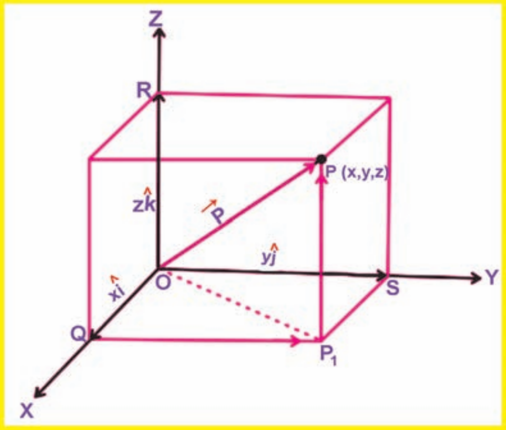


Figure 10.

As shown in the figure, Let P_1 be the foot of the perpendicular from point P on the plane XOY. Observe that P_1P is parallel to the z-axis. We know that, $\vec{i}, \vec{j}$, and $\vec{k}$ are unit vectors along the x, y, and z-axes, respectively. Hence, by definition of the coordinates of point P, we have

$$\overrightarrow{P_1P} = \overrightarrow{OR} = z\vec{k}$$

Similarly, we have

$$\overrightarrow{QP_1} = \overrightarrow{OS} = y\vec{j} \text{ and}$$

$$\overrightarrow{P_1S} = \overrightarrow{OQ} = x\vec{i}$$

Now, by using the triangle law of vector addition, we can write

$$\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\vec{i} + y\vec{j}$$

And,

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\vec{i} + y\vec{j} + z\vec{k}$$

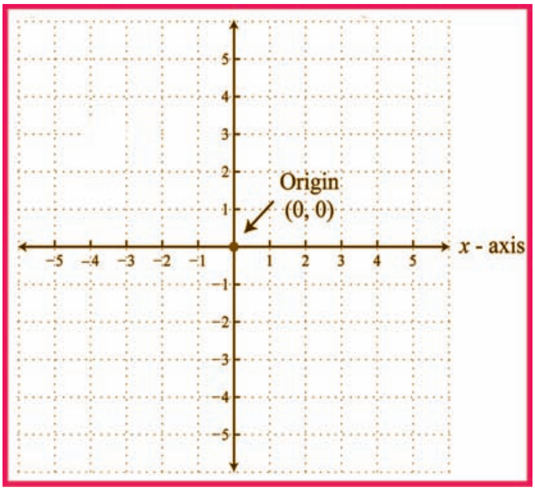
Therefore, the position vector of P with reference to O is

$$\overrightarrow{OP} \text{ (or } \vec{r}) = x\vec{i} + y\vec{j} + z\vec{k}$$

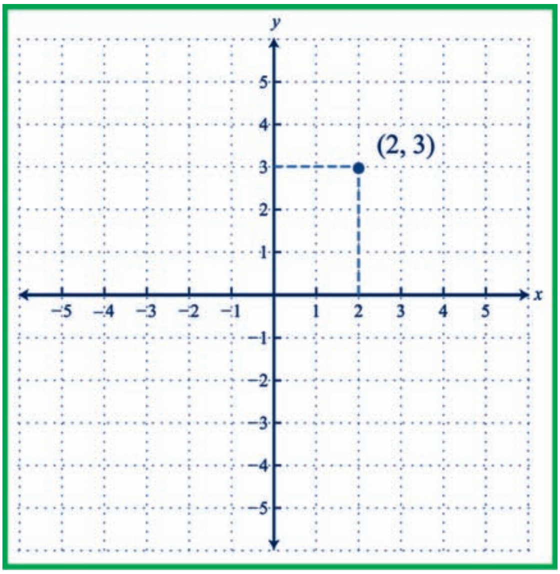
This is the Component Form of a vector. Here, x, y, and z are the scalar components of $\vec{r}$ and $x\vec{i}$, $y\vec{j}$, and $z\vec{k}$ are the vector components of $\vec{r}$ along the respective axes. The scalar components are also referred to as rectangular components at times.

1.4 RECTANGULAR COORDINATE SYSTEM

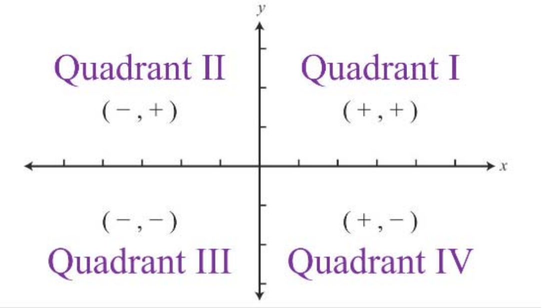
The rectangular coordinate system consists of two real number lines that intersect at a right angle. The horizontal number line is called the x-axis, and the vertical number line is called the y-axis. These two number lines define a flat surface called a plane, and each point on this plane is associated with an ordered pair of real numbers (x, y). The first number is called the x-coordinate, and the second number is called the y-coordinate. The intersection of the two axes is known as the origin, which corresponds to the point (0, 0).



An ordered pair (*x*, *y*) represents the position of a point relative to the origin. The *x*-coordinate represents a position to the right of the origin if it is positive and to the left of the origin if it is negative. The *y*-coordinate represents a position above the origin if it is positive and below the origin if it is negative. Using this system, every position (point) in the plane is uniquely identified. For example, the pair (2, 3) denotes the position relative to the origin as shown:



This system is often called the Cartesian coordinate system, named after the French mathematician René Descartes (1596–1650). The *x*- and *y*-axes break the plane into four regions called quadrants, named using roman numerals I, II, III, and IV, as pictured. In quadrant I, both coordinates are positive. In quadrant II, the *x*-coordinate is negative and the *y*-coordinate is positive. In quadrant III, both coordinates are negative. In quadrant IV, the *x*-coordinate is positive and the *y*-coordinate is negative.



1.5 THE DOT PRODUCT

The dot product of two vectors a and b (also called the scalar product) is denoted by $a \cdot b$. It is a scalar defined by

$a \cdot b = |a|\ |b| \cos\theta$.

θ here is the angle between the vectors when their initial points coincide and is restricted to the range $0 \le \theta \le \pi$, Figure 11.

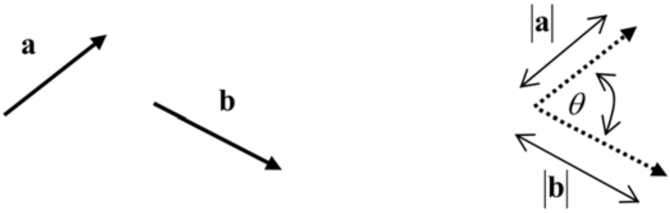


Figure 11: the dot product.

An important property of the dot product is that if for two (proper) vectors a and b, the relation $a \cdot b = 0$, then a and b are perpendicular. The two vectors are said to be orthogonal. Also, $a \cdot a = |a|\ |a| \cos(0)$, so that the length of a vector is $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

Another important property is that the projection of a vector u along the direction of a unit vector e is given by $u \cdot e$. This can be interpreted geometrically as in Figure 12.

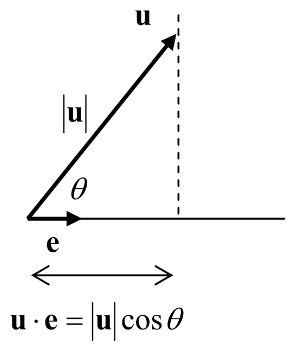


Figure 12: the projection of a vector along the direction of a unit vector.

It follows that any vector u can be decomposed into a component parallel to a (unit) vector e and another component perpendicular to e, according to

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{e})\mathbf{e} + [\mathbf{u} - (\mathbf{u}\cdot\mathbf{e})\mathbf{e}]$$

1.6 THE CROSS PRODUCT

The cross product of two vectors a and b (also called the vector product) is denoted by a ×b. It is a vector with magnitude

$|a\times b| = |a|\ |b|\ \sin\theta$

with θ defined as for the dot product. It can be seen from the figure that the magnitude of a ×b is equivalent to the area of the parallelogram determined by the two vectors a and b.

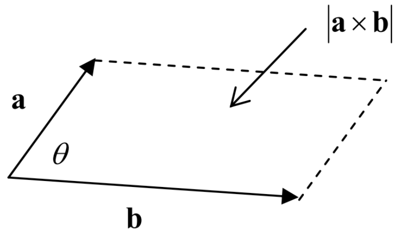


Figure 13: The magnitude of the cross product.

The direction of this new vector is perpendicular to both a and b. Whether a ×b points "up" or "down" is determined from the fact that the three vectors a, b and a ×b form a right handed system. This means that if the thumb of the right hand is pointed in the direction of a ×b , and the open hand is directed in the direction of a, then the curling of the fingers of the right hand so that it closes should move the fingers through the angle θ, $0 \le \theta \le \pi$, bringing them to b. Some examples are shown in Figure 14.

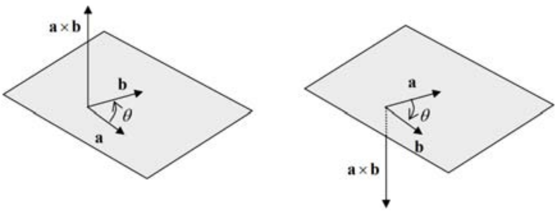


Figure 14: examples of the cross product.

The cross product possesses the following properties (which can be proved using the above definition):

(1) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ (*not* commutative)
(2) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (distributive)
(3) $\alpha(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(\alpha\mathbf{b})$
(4) $\mathbf{a}\times\mathbf{b} = \mathbf{o}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ $(\neq \mathbf{o})$ are parallel ("linearly dependent")

1.6.1 The Triple Scalar Product

The triple scalar product, or box product, of three vectors u, v, w is defined by

$$\boxed{(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}}$$

Its importance lies in the fact that, if the three vectors form a right-handed triad, then the volume V of a parallelepiped spanned by the three vectors is equal to the box product.

To see this, let e be a unit vector in the direction of $\mathbf{u}\times\mathbf{v}$, Figure 15. Then the projection of w on $\mathbf{u}\times\mathbf{v}$ is $h = \mathbf{w}\cdot\mathbf{e}$, and

$$\begin{aligned}\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) &= \mathbf{w}\cdot(|\mathbf{u}\times\mathbf{v}|\mathbf{e}) \\ &= |\mathbf{u}\times\mathbf{v}|h \\ &= V\end{aligned}$$

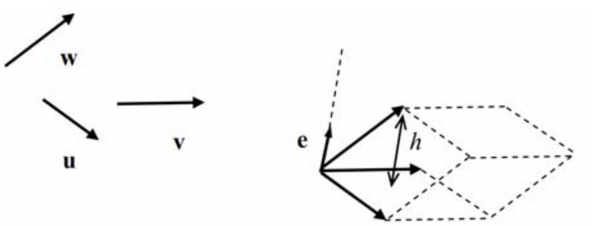


Figure 15: the triple scalar product.