ENGINEERING TRIGONOMETRY

PEASE
and
WADSWORTH

CHAPTER 1

TRIGONOMETRY DEFINED. VOCABULARY BUILD-UP

1. *What is the content of a Trigonometry course?*
2. *What facts from previous courses will be frequently needed in studying Trigonometry?*

1-1. Scope of Trigonometry.—The word *trigonometry* means, literally, the measurement of triangles. It may be interpreted more broadly as that branch of mathematics wherein we evaluate the magnitudes of the sides and angles of triangles and solve problems which ultimately involve triangles.

1-2. Some Prerequisites to the Study of Trigonometry.—The purpose of this article is to refresh the students' minds on the following terms: angle, positive angle, negative angle, acute angle, obtuse angle, initial side, terminal side, ordinate and abscissa.

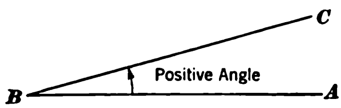


FIG. 1-1

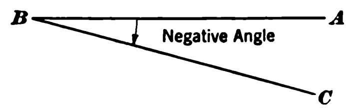


FIG. 1-2

In Fig. 1-1, we have two lines, BA and BC. If they are initially superimposed one on the other, and if BA is held stationary while BC is rotated counter-clockwise to some new position, the opening so formed is termed a *positive angle.* If the line is rotated clockwise, as shown in Fig. 1-2, the angle formed is *negative.* Side BA is known as the *initial side,* and side BC as the *terminal side,* in both cases.

The terms *acute* for an angle less than 90° and *obtuse* for an angle greater than 90° are used just as in plane geometry. At this time, however, it would prove advantageous for students to fully realize that an angle of any size, even greater than 360°, may be drawn by continuing to revolve the terminal side until the desired number of degrees have been generated. For example, for a 540° angle, one would revolve the terminal side counterclockwise through one complete revolution and an additional half-revolution, as shown in Fig. 1-3.

Obviously, any angle might be placed with its initial side on OX or elsewhere. When the initial side is on OX, the angle is said to be *conventionally located.*

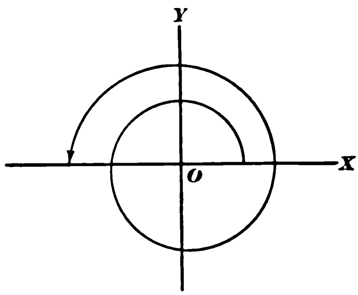


FIG. 1-3

It is assumed that the students know what a degree is and that there are 90 degrees in a right angle and 360 degrees in one revolution. Furthermore, an angle of 1 degree can be divided into 60 equal parts, each of which is 1 *minute.* A minute, in turn, can be subdivided into 60 equal parts, each of which is 1 *second.*

But another unit of angular measure, the *radian,* may not be so well known. If we have a central angle, the sides of which intercept a portion of the circumference of a circle equal in length to the radius of the circle, the measure of this central angle is defined as a *radian.* From geometry we know that the circumference of a circle is equal to 2π (pronounced *pie*) times the radius. In other words, the radius can be marked off on the circumference exactly 2π times. Therefore, there are 2π radians in the central angle subtended by the complete circumference. By dividing 360° by 2π, we find

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

or

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian}$$

Numerically, 1 radian $= 57.2958° = 57°17'45''$, approximately.

Also, $180° = \pi$ radians, and $90° = \frac{\pi}{2}$ radians.

Many applications of radian measure can be built around the formula

$$S = r\theta \tag{1}$$

where r represents the radius of a circle measured in any convenient linear unit; θ (pronounced *theta*) is the central angle, which must be expressed in radians and cannot be expressed in degrees; and S is the arc of the circumference expressed in the same unit as r.

Formula 1 can be justified in the following manner: In a circle with any radius, r, the entire circumference, C, is to the complete central angle, 2π radians, as any portion, S, of the circumference is to its central angle, θ. That is,

$$\frac{C}{2\pi}=\frac{S}{\theta} \tag{2}$$

But, $C=2\pi r$. Substituting in formula 2, we obtain formula 1. In other words, in a circle of radius 10 feet, a 90° angle $\left(\frac{\pi}{2}\text{ radians}\right)$ will intercept an arc of length

$$S=10\times\frac{\pi}{2}=5\pi \text{ feet}$$

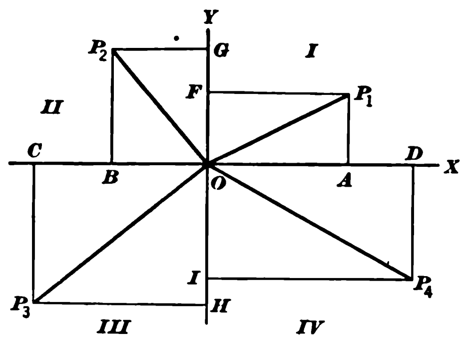


FIG. 1-4

Fig. 1-4 is introduced to illustrate a coordinate system. OX and OY are two perpendicular lines. These lines divide the plane into four *quadrants*, I, II, III and IV. OX is termed the *X-axis*, OY the *Y-axis*, and O the *origin*. Let P_1, P_2, P_3 and P_4 represent any points on the plane. Then P_1A, P_2B, P_3C and P_4D, the perpendicular distances to the X-axis, are the *ordinates*. An ordinate is *positive* when the point is above the X-axis, as is P_1 or P_2, and is *negative* when the point is below the X-axis, as is P_3 or P_4. Also, P_1F, P_2G, P_3H and P_4I, the perpendicular distances to the Y-axis, are termed *abscissas*. When the point is to the right of the Y-axis, as is P_1 or P_4, the abscissa is *positive*; and, when the point is to the left, the abscissa is *negative*. The fact that the signs are so taken is merely a matter of convention.

The radial lines OP_1, OP_2, OP_3 and OP_4, Fig. 1-4, are called *radii* or *distances*, and *they are always positive*. The reason why radial distances are always considered positive is not immediately evident, but the student is urged to retain this fact, as it will prove absolutely necessary later. We frequently shall be treating the terminal side of an angle as a radial line and, hence, shall be considering this side positive. A short way of designating a point's location is (abscissa, ordinate). For instance, (2, 3) is a point 2 units to the right of the Y-axis, and 3 units up from the X-axis. Note that this whole *rectangular system* is merely a scheme used to convey a point from one mind to a second mind. Other schemes where the axes are not at right angles could also be used. Such schemes would be named "coordinate systems" but not "rectangular coordinate systems."

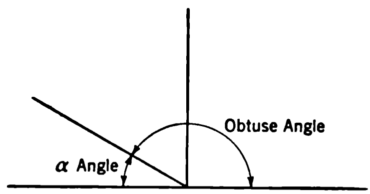


FIG. 1-5

We can now define the "α-angle," called the alpha angle. Any time an angle greater than 90° is conventionally located, an acute angle is formed between the terminal side and the X-axis, as is shown in Fig. 1-5. For future convenience, the authors wish to name this acute angle the "α-angle." For instance, the "α-angle" of 160° is 20°, the "α-angle" of 225° is 45°, etc. By definition allow the "α-angle" to be *positive* and *acute*. This "α-angle" is sometimes termed the "reference angle" or "related angle."

QUESTIONS

In questions 1 to 6, inclusive, give your conception of the terms mentioned.

1. Trigonometry, angle, initial side, terminal side.
2. Positive angle, negative angle, acute angle, obtuse angle.
3. Coordinates, coordinate system, rectangular coordinate system.
4. Axes, abscissa, ordinate, radial distance.
5. Clockwise, counter-clockwise, quadrant.
6. An angle greater than 360°.

7. If P is any point on the plane of the axes OX and OY, name each of the following lines: (*a*) the line joining the origin to the point P; (*b*) the line from P perpendicular to the X-axis; (*c*) the line from P perpendicular to the Y-axis.

8. If P is above the X-axis and to the right of the Y-axis, answer the following questions: (*a*) Would the abscissa and ordinate of P be positive or negative? (*b*) Would the radial distance be positive or negative? (*c*) In what quadrant would P be found?

9. Name the quadrant in which the terminal side of each of the following angles appears: (*a*) 120°; (*b*) −150°; (*c*) 80°; (*d*) 330°.

10. Determine the "α-angle" of the following: (*a*) 150°; (*b*) 330°; (*c*) −150°.

11. Determine the sign of the radial distance to a point: (*a*) in the third quadrant; (*b*) in any quadrant.

12. (*a*) Is the alpha angle always positive? (*b*) Is it always acute?

CONVENTIONAL PROBLEMS

Note: Give answers in terms of π if convenient. Superscript r means radians.

1. Draw the following angles, showing initial and terminal sides and direction of rotation: an acute angle, 180°, $-270°$, $\frac{5\pi^r}{4}$, $-\frac{3\pi^r}{4}$.

2. A man turns a crank through two complete revolutions. Through how many radians does the crank turn? Repeat for a crank 2 in. longer.

3. State the relation which is convenient in changing a given number of: (*a*) degrees to radians; (*b*) radians to degrees.

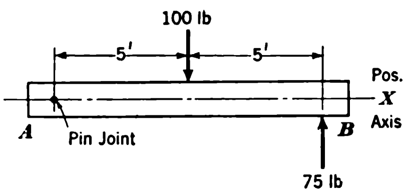
4. Express in degrees: 3^r, $\frac{\pi^r}{2}$, $-2\pi^r$.

5. Express in radians: 30°, 45°, 60°, 90°, 180°.

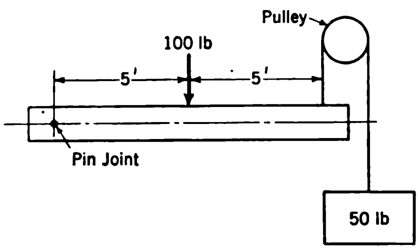
6. Express 30°20′10″ in: (*a*) minutes; (*b*) seconds.

7. What is the magnitude and sense of the angle generated by the minute hand of a clock between 1 A.M. and 3 A.M. (sense refers to clockwise or counter-clockwise)?

8. A wheel axle moves 16 ft per second parallel to a plane along which the wheel runs. Express the angular velocity of the wheel in radians per minute, assuming that the radius of the wheel is 3 ft.



PROBLEM 11



PROBLEM 12

9. Move one cent around a stationary cent, the two coins touching but not slipping. Through how many radians around its own axis does the moving cent turn?

10. 2π ft of circular arc subtends an angle of 1′ at its center. What is the radius?

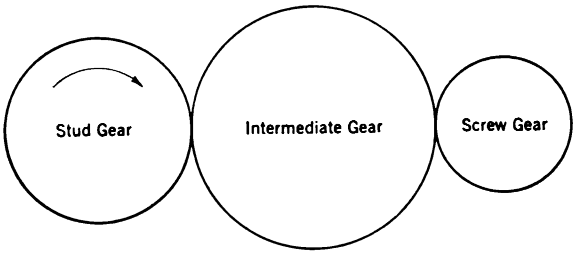
11. (*a*) Will the beam *AB* rotate clockwise or counter-clockwise? (*b*) Will a positive or negative angle be generated? Use high-school physics.

12. Repeat Problem 11 for the beam in the illustration.

ENGINEERING APPLICATIONS

13. On most machine tools there is a train of three gears like that in the illustration. The stud gear has 50 teeth and is rotating clockwise at 100 rpm. If the intermediate gear has 100 teeth and the screw gear has 25 teeth, answer the following questions: (*a*) Through how many radians per minute does the intermediate gear turn and in what direction does the rotation occur? (*b*) Through how many radians per minute and in what direction does the screw gear rotate? (*c*) Suppose that the stud gear is connected directly to the screw gear. If the direction and speed of the stud gear remain as before, through how many radians per minute will the screw gear turn?

14. A pulley of 5 in. radius is driven in a clockwise direction by a belt. (*a*) If every point on the belt moves through a distance of 20π in., through how many radians would the pulley rotate? (*b*) If the pulley had twice the radius, through how many degrees would it move?



PROBLEM 13

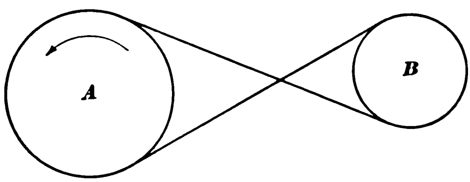
15. The typical $\frac{1}{4}$- or $\frac{1}{8}$-horsepower alternating-current motor adapted for household use generally has a speed of 1,800 rpm. Compute its speed in radians per second.

16. The speed (rpm) of an alternating-current motor depends on the frequency, f, of the current supplied to the motor and on the number of poles, P, wound into the field of the motor. Expressed symbolically, the speed is

$$S = \frac{120f}{P}$$

Suppose a motor designed for a frequency of 25 cycles per second and a speed of 1,500 rpm is deliberately supplied with 60-cycle current. Compute its speed in radians per second.

17. Using the information in the preceding example, suppose that we wish to change the number of poles in the 25-cycle, 1500-rpm motor so that we can continue to run it at $f = 25$, but at a new speed of $1{,}000\pi$ radians per minute. How many poles should we build into the motor?



PROBLEM 18

18. In the illustration, pulley A (radius = 10 in.) is rotating counterclockwise at 100 rpm. Determine the speed of pulley B in radians per second and the sense of rotation. Pulley B has a radius of 5 in.

19. In passing through a dye vat, a cloth travels over a roller 1 ft in diameter and then over another roller 2 ft in diameter. If 20 ft of cloth pass through the machine, through how many radians will each roller turn?

CHAPTER 2

TRIGONOMETRIC FUNCTIONS. RIGHT TRIANGLES

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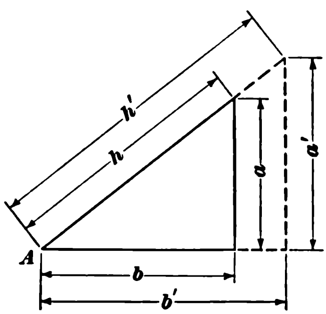
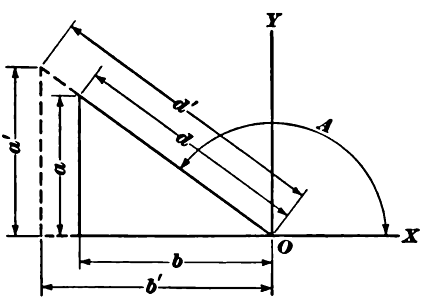
1. *What functions occur so often in Trigonometry that they are referred to as Trigonometric Functions?*
2. *How are these functions defined and interrelated?*
3. *How are the numerical values of these functions determined?*
4. *What is meant by "solving a triangle"?*
5. *How are the Trigonometric Functions used in solving a right triangle?*
6. *What do the graphs of these functions look like, and what information do the graphs supply?*

2-1. Definitions and Interrelations of the Trigonometric Functions of an Angle.—The trigonometric functions of an angle



Right-Triangle Viewpoint

FIG. 2-1



General-Angle Viewpoint

FIG. 2-2

are ratios. Hence, we will refer to them as trigonometric ratios. For "ratio" the usual definition holds; i.e., a ratio is the quotient of two quantities, such as $\frac{A}{B}$. Experience justifies defining each of the trigonometric functions from two viewpoints: the right-triangle viewpoint, and the general-angle viewpoint. The latter proves more convenient than the former in certain instances, and vice versa. The six functions, sine (sin), cosine (cos), tangent (tan), cotangent (cot or ctn), secant (sec), and cosecant (csc), are defined below for any angle A in terms of the algebraic quantities shown in Figs. 2-1 and 2-2. In every case, h and d are considered positive.

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse*}} = \frac{a}{h} = \frac{a'}{h'} \text{ or } \frac{\text{ordinate}}{\text{distance}} = \frac{a}{d} = \frac{a'}{d'}$$

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{b}{h} = \frac{b'}{h'} \text{ or } \frac{\text{abscissa}}{\text{distance}} = \frac{b}{d} = \frac{b'}{d'}$$

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{b} = \frac{a'}{b'} \text{ or } \frac{\text{ordinate}}{\text{abscissa}} = \frac{a}{b} = \frac{a'}{b'}$$

$$\cot A = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{b}{a} = \frac{b'}{a'} \text{ or } \frac{\text{abscissa}}{\text{ordinate}} = \frac{b}{a} = \frac{b'}{a'}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{h}{b} = \frac{h'}{b'} \text{ or } \frac{\text{distance}}{\text{abscissa}} = \frac{d}{b} = \frac{d'}{b'}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{h}{a} = \frac{h'}{a'} \text{ or } \frac{\text{distance}}{\text{ordinate}} = \frac{d}{a} = \frac{d'}{a'}$$

Note that extending the sides of angle A, as shown by the dotted extensions, causes $\frac{a}{h}$ to become $\frac{a'}{h'}$. But $\frac{a}{h} = \frac{a'}{h'}$. Hence, $\sin A$ remains unchanged in spite of the change of length of the sides. Analogous statements could be made for the remaining five trigonometric ratios.

The pairs of functions connected by brackets are mutually reciprocal; i.e.,

$$\csc A = \frac{1}{\sin A} \text{ or } \sin A = \frac{1}{\csc A}$$

$$\sec A = \frac{1}{\cos A} \text{ or } \cos A = \frac{1}{\sec A}$$

$$\cot A = \frac{1}{\tan A} \text{ or } \tan A = \frac{1}{\cot A}$$

* Hypotenuse is that side of a right triangle which is opposite the right angle.

The foregoing relations follow from the definitions. By the definitions and division, it may be shown that

$$\tan A = \frac{\sin A}{\cos A} \text{ and } \cot A = \frac{\cos A}{\sin A}$$

Also, by the definitions and the Pythagorean relation,* we have

$$\sin^2 A + \cos^2 A = 1$$

where $\sin^2 A$ means $(\sin A)^2$ and $\cos^2 A$ means $(\cos A)^2$.

Just as a secretary uses a certain character to represent a stock phrase, such as "Dear Sir," so too the mathematician uses a negative 1 put in a position normally occupied by an exponent. Thus, $\sin^{-1} 0.5$ is translated "the angle whose sine is 0.5." Again, $\cos^{-1} 0.5$ means "the angle whose cosine is 0.5." This symbol $^{-1}$ is not an exponent and does not obey the laws of exponents. It is merely used for brevity. The "α-angle" was defined as positive and acute. Hence, any trigonometric ratio of any "α-angle" will always be positive.

2-2. Numerical Values of the Trigonometric Ratios for the Special Angles 30°, 45°, 60° and Multiples of 90°.—A spontaneous knowledge of the numerical values of the trigonometric ratios for the special angles 30°, 60°, and 45° is most desirable. The ability to visualize a triangle, which may be drawn in the following manner, will help in this memory task. Draw a triangle ABC,

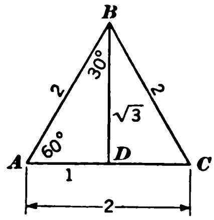


FIG. 2-3

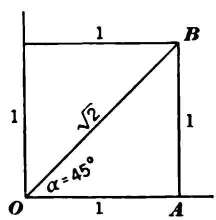


FIG. 2-4

Fig. 2-3, each side of which is 2 units long. Drop a perpendicular BD from any vertex to the opposite side, thus forming a triangle ABD having sides $AB=2$, $AD=1$, and (by the Pythagorean relation) $BD=\sqrt{3}$. The angles of triangle ABD are: $A=60°$, $D=90°$, and $B=30°$. Hence, using the definitions of sine and cosine, we have: $\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$, and $\cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$.

* The Pythagorean relation is: "In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides."

In other words, by being able to summon a mental picture of the finished triangle ABD, one can obtain quickly any trigonometric ratio of the special angle 30° or 60°, without resorting to tables.

In the case of the 45° angle, locate a unit square on two rectangular axes, as shown in Fig. 2-4. Connect O to B, thus completing the triangle OAB. Angle α is a 45° angle and OB (by the Pythagorean relation) is $\sqrt{2}$ units long. Now, by the right-triangle definitions, $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, and similar ratios may be written for the remaining trigonometric ratios.

Other angles which also deserve special consideration are the multiples of 90°, such as 0°, 90°, 180°, 270°, 360°, 450°, etc. To obtain a trigonometric ratio for one of these angles, proceed as follows:

Step 1. Draw an angle which approximates in magnitude the angle under discussion and set up the fraction expressing the desired trigonometric ratio in literal form (by use of letters).

Step 2. Observe the manner in which the numerator, the denominator, or both the numerator and denominator of the literal ratio vary as you mentally allow the approximate angle to approach more nearly in magnitude the angle under discussion.

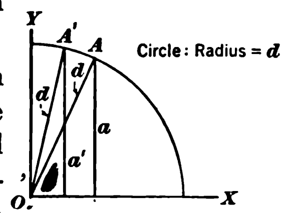


FIG. 2-5

Step 3. Conclude what the literal ratio is ultimately approaching, basing your decision on your observation.

ILLUSTRATIVE EXAMPLE 1

STATEMENT.—Evaluate the ratio sin 90°.

SOLUTION.—*Step 1:* As shown in Fig. 2-5, we draw an angle XOA fairly close to 90° and set up the fraction for the desired trigonometric ratio in literal form as $\sin XOA = \frac{a}{d}$.

Step 2: As the angle XOA grows and becomes XOA', thus approaching the angle 90°, we observe that a becomes a', thus becoming more nearly equal to the denominator d. In fact we can make a become as nearly equal to d as we please by causing the approximate angle to approach sufficiently close to 90°.

Step 3: In consequence of the observation in Step 2, we conclude that $\frac{a}{d}$ approaches $\frac{d}{d}$, which is equal to 1. Therefore, we conclude that $\sin 90° = 1$.

ILLUSTRATIVE EXAMPLE 2

STATEMENT.—Evaluate the ratio tan 90°.

SOLUTION.—*Step 1:* As shown in Fig. 2-6, we draw an angle XOA approximately equal to 90°, and set up the fraction for the tangent in literal form as $\tan XOA = \frac{a}{b}$.

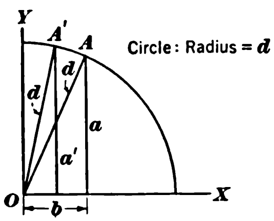


FIG. 2-6

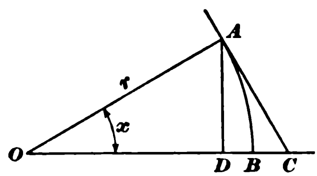


FIG. 2-7

Step 2: As the angle XOA grows and becomes XOA', thus approaching 90°, we observe that b is approaching zero, and that a is approaching d in length. In fact, we can make b as near to zero as we wish by letting XOA approach close enough to 90°.

Step 3: In consequence of the observations in Step 2, we conclude that $\frac{a}{b}$ approaches $\frac{d}{0}$. Therefore, we write $\tan 90° = \infty$, which is read "tan 90° is infinite." When saying $\tan 90° = \infty$, we mean two things:

1. As the angle approaches 90° from the first quadrant, the tangent of the angle increases without bound.

2. When the angle *is* 90°, the fraction for the tangent involves division by zero, which is not accepted in algebra. Hence, the tangent of 90° is non-existent as far as finite values are concerned.

At this point the student is urged to note the following conditions: As the angle approaches 90° from the first quadrant, the tangent of the angle increases without limit through positive values; whereas, as the angle approaches 90° from the second quadrant, the tangent increases without limit through negative values. In the former case, the abscissa b is positive; whereas, in the latter case, b is negative.

ILLUSTRATIVE EXAMPLE 3

STATEMENT.—Find the limit* of $\frac{\sin x}{x}$ as x approaches zero, where x is expressed in radians.

SOLUTION.—In Fig. 2-7, angle x is close to zero and measured in radians; OAB is a circular sector with any convenient radius r; AC is a tangent to the arc AB at A; and AD is a perpendicular from A to OB. By inspection of the diagram,

$$\begin{matrix}\text{Area of} \\ \text{triangle } OAD\end{matrix} < \begin{matrix}\text{Area of} \\ \text{sector } OAB\end{matrix} < \begin{matrix}\text{Area of} \\ \text{triangle } OAC\end{matrix} \quad (1)$$

But, by plane geometry, area of $OAD = \frac{1}{2}(OD)(AD) = \frac{1}{2}(r \cos x)(r \sin x)$, and area of sector $OAB = \frac{1}{2}r^2x$. Also, area of $OAC = \frac{1}{2}(OA)(AC) = \frac{1}{2}r(r \tan x)$.

Substituting in equation 1 and dividing by $\frac{1}{2}r^2 \sin x$, we have

$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or

$$\frac{1}{\cos x} > \frac{\sin x}{x} > \cos x$$

But, by the three-step process, $\cos x$ approaches 1 as x approaches zero. Therefore, $\frac{\sin x}{x}$ always lies between two quantities both of which are approaching 1. Hence, we conclude that the limit of $\frac{\sin x}{x}$ is 1.

Also it follows that $\sin x = \tan x = x$, approximately, when x, measured in radians, is a sufficiently small angle.

QUESTIONS

1. Using just one word, what are trigonometric functions?
2. From what two viewpoints has the text defined trigonometric functions? Elaborate. Are the terms abscissa and ordinate used in the right-

* A variable x is said to approach a constant "a" as a limit if the absolute value of $(a-x)$ becomes and remains less than any arbitrarily small preassigned positive quantity V when the variable x takes all values for which it is defined in the neighborhood of "a".

triangle viewpoint or the general-angle viewpoint? What are the corresponding terms in the right-triangle viewpoint?

3. State several reciprocal relations between the trigonometric functions. State the relation between the sine, cosine, and tangent of any angle.

4. In obtaining the values of the trigonometric ratios for the special angles 30°, 60° and 45°, without resorting to tables, special triangles prove advantageous. Describe the construction of the triangles and the methods used to get the functional values.

5. What angles other than 30°, 60° and 45° also come under the head of *special angles?* How does one determine the trigonometric ratios of these angles?

6. What is the value of $\lim_{x \to 0} \frac{\sin x}{x}$, $\lim_{x \to 0} \frac{\tan x}{x}$? What unit must be used in measuring x? Define limit.

7. When does $\sin \theta = \tan \theta = \theta$? In what units must θ be measured?

8. What is the meaning of "infinity" when applied to tan 90°?

9. If $\text{vers } x = 1 - \cos x$ and $\text{hav } x = \frac{1}{2} \text{ vers } x = \frac{1}{2}(1 - \cos x)$, evaluate the versed sine or versine (vers) and the haversed sine or haversine (hav) of 30°, 45°, 60°, 90°, 180°, 270°, and 360°.

10. Is every trigonometric ratio of every α-angle positive?

2-3. General Procedure for Solving a Right Triangle.—The definitions and facts so far discussed are sufficient for solving a right triangle. A right triangle consists of six parts; namely, a right angle, two acute angles, and three sides. In the statement of a problem we are given the right angle and the values of two additional parts. One of the two additional parts must be a side if we are to avoid the case where three angles are given. This case does not yield a unique (one and only one) triangle. By "solving the triangle," we mean finding the lengths of the remaining sides and the number of degrees in each of the remaining angles.

The definition of any one of the six trigonometric ratios of an angle can be treated as a formula connecting three variables. Hence, if in a formula such as $\sin A = \frac{a}{h}$ we have any two of the three quantities, we can solve for the third. It proves convenient to designate any angle and the opposite side by one and the same letter, the capital notation being used for the angle. When it is possible without too much inconvenience, use given data in preference to computed values. For, should these computed values be wrong, then using them would result in an error in the next part also, even though your workmanship should prove flawless in the latter computations.