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ELECTRICAL ENGINEER'S REFERENCE BOOK

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Units, Mathematics and Physical Quantities

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(Section 1.2.10)

Contents

This reference section provides (a) a statement of the International System (SI) of Units, with conversion factors; (b) basic mathematical functions, series and tables; and (c) some physical properties of materials.

1.1 International unit system

The International System of Units (SI) is a metric system giving a fully coherent set of units for science, technology and engineering, involving no conversion factors. The starting point is the selection and definition of a minimum set of independent 'base' units. From these, 'derived' units are obtained by forming products or quotients in various combinations, again without numerical factors. For convenience, certain combinations are given shortened names. A single SI unit of energy (joule = kilogram metre-squared per second-squared) is, for example, applied to energy of any kind, whether it be kinetic, potential, electrical, thermal, chemical . . . , thus unifying usage throughout science and technology.

The SI system has seven *base* units, and two *supplementary* units of angle. Combinations of these are *derived* for all other units. Each physical quantity has a quantity symbol (e.g. *m* for mass, *P* for power) that represents it in physical equations, and a unit symbol (e.g. kg for kilogram, W for watt) to indicate its SI unit of measure.

1.1.1 Base units

Definitions of the seven base units have been laid down in the following terms. The quantity symbol is given in italic, the unit symbol (with its standard abbreviation) in roman type. As measurements become more precise, changes are occasionally made in the definitions.

Length: l, metre (m) The metre was defined in 1983 as the length of the path travelled by light in a vacuum during a time interval of 1/299 792 458 of a second.

Mass: m, kilogram (kg) The mass of the international prototype (a block of platinum preserved at the International Bureau of Weights and Measures, Sèvres).

Time: t, second (s) The duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

Electric current: i, ampere (A) The current which, maintained in two straight parallel conductors of infinite length, of negligible circular cross-section and 1 m apart in vacuum, produces a force equal to 2×10^{-7} newton per metre of length.

Thermodynamic temperature: T, kelvin (K) The fraction 1/273.16 of the thermodynamic (absolute) temperature of the triple point of water.

Luminous intensity: I, candela (cd) The luminous intensity in the perpendicular direction of a surface of 1/600 000 m^2 of a black body at the temperature of freezing platinum under a pressure of 101 325 newton per square metre.

Amount of substance: Q, mole (mol) The amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kg of carbon-12. The elementary entity must be specified and may be an atom, a molecule, an ion, an electron . . . , or a specified group of such entities.

1.1.2 Supplementary units

Plane angle: α, β, . . ., radian (rad) The plane angle between two radii of a circle which cut off on the circumference of the circle an arc of length equal to the radius.

Solid angle: Ω, steradian (sr) The solid angle which, having its vertex at the centre of a sphere, cuts off an area of the surface of the sphere equal to a square having sides equal to the radius.

1.1.3 Notes

Temperature At zero K, bodies possess no thermal energy. Specified points (273.16 and 373.16 K) define the Celsius (centigrade) scale (0 and 100°C). In terms of *intervals*, 1°C = 1 K. In terms of *levels*, a scale Celsius temperature θ corresponds to (θ + 273.16) K.

Force The SI unit is the newton (N). A force of 1 N endows a mass of 1 kg with an acceleration of 1 m/s^2.

Weight The weight of a mass depends on gravitational effect. The standard weight of a mass of 1 kg at the surface of the earth is 9.807 N.

1.1.4 Derived units

All physical quantities have units derived from the base and supplementary SI units, and some of them have been given names for convenience in use. A tabulation of those of interest in electrical technology is appended to the list in *Table 1.1*.

Table 1.1 SI base, supplementary and derived units

Quantity	*Unit name*	*Derivation*	*Unit symbol*
Length	metre		m
Mass	kilogram		kg
Time	second		s
Electric current	ampere		A
Thermodynamic temperature	kelvin		K
Luminous intensity	candela		cd
Amount of substance	mole		mol
Plane angle	radian		rad
Solid angle	steradian		sr
Force	newton	kg m/s^2	N
Pressure, stress	pascal	N/m^2	Pa
Energy	joule	N m, W s	J
Power	watt	J/s	W
Electric charge, flux	coulomb	A s	C
Magnetic flux	weber	V s	Wb
Electric potential	volt	J/C	V
Magnetic flux density	tesla	Wb/m^2	T
Resistance	ohm	V/A	Ω
Inductance	henry	Wb/A, V s/A	H
Capacitance	farad	C/V, A s/V	F
Conductance	siemens	A/V	S
Frequency	hertz	s^{-1}	Hz
Luminous flux	lumen	cd sr	lm
Illuminance	lux	lm/m^2	lx
Radiation activity	becquerel	s^{-1}	Bq
Absorbed dose	gray	J/kg	Gy
Mass density	kilogram per cubic metre		kg/m^3
Dynamic viscosity	pascal-second		Pa s
Concentration	mole per cubic metre		mol/m^3
Linear velocity	metre per second		m/s
Linear acceleration	metre per second-squared		m/s^2
Angular velocity	radian per second		rad/s

cont'd

Table 1.1 (*continued*)

Quantity	*Unit name*	*Derivation*	*Unit symbol*
Angular acceleration	radian per second-squared		rad/s^2
Torque	newton metre		N m
Electric field strength	volt per metre		V/m
Magnetic field strength	ampere per metre		A/m
Current density	ampere per square metre		A/m^2
Resistivity	ohm metre		Ω m
Conductivity	siemens per metre		S/m
Permeability	henry per metre		H/m
Permittivity	farad per metre		F/m
Thermal capacity	joule per kelvin		J/K
Specific heat capacity	joule per kilogram kelvin		J/(kg K)
Thermal conductivity	watt per metre kelvin		W/(m K)
Luminance	candela per square metre		cd/m^2

Decimal multiples and submultiples of SI units are indicated by prefix letters as listed in *Table 1.2*. Thus, kA is the unit symbol for kiloampere, and μF that for microfarad. There is a preference in technology for steps of 10^3. Prefixes for the kilogram are expressed in terms of the gram: thus, 1000 kg = 1 Mg, not 1 kkg.

1.1.5 Auxiliary units

Some quantities are still used in special fields (such as vacuum physics, irradiation, etc.) having non-SI units. Some of these are given in *Table 1.3* with their SI equivalents.

1.1.6 Conversion factors

Imperial and other non-SI units still in use are listed in *Table 1.4*, expressed in the most convenient multiples or sub-multiples of the basic SI unit [ ] under classified headings.

1.1.7 CGS electrostatic and electromagnetic units

Although obsolescent, electrostatic and electromagnetic units (e.s.u., e.m.u.) appear in older works of reference. Neither system is 'rationalised', nor are the two mutually compatible. In e.s.u. the electric space constant is $\varepsilon_0 = 1$, in e.m.u. the magnetic space constant is $\mu_0 = 1$; but the SI units take account of the fact that $1/\sqrt{(\varepsilon_0\mu_0)}$ is the velocity of electromagnetic wave propagation in free space. *Table 1.5* lists SI units with the equivalent number *n* of e.s.u. and e.m.u. Where these lack names, they are expressed as SI unit names with the prefix 'st' ('electrostatic') for e.s.u. and 'ab' ('absolute') for e.m.u. Thus, 1 V corresponds to $10^{-2}/3$ stV and to 10^8 abV, so that 1 stV = 300 V and 1 abV = 10^{-8} V.

1.2 Mathematics

Mathematical symbolism is set out in *Table 1.6*. This sub-section gives trigonometric and hyperbolic relations, series (including Fourier series for a number of common wave forms), binary enumeration and a list of common derivatives and integrals.

Table 1.2 Decimal prefixes

10^{18} exa E	10^9 giga G	10^2 hecto h	10^{-3} milli m	10^{-12} pico p
10^{15} peta P	10^6 mega M	10^1 deca da	10^{-6} micro μς	10^{-15} femto f
10^{12} tera T	10^3 kilo k	10^{-1} deci d	10^{-9} nano n	10^{-18} atto a
		10^{-2} centi c		

Table 1.3 Auxiliary units

Quantity	*Symbol*	*SI*		*Quantity*	*Symbol*	*SI*	
Angle				Mass			
degree	(°)	π/180	rad	tonne	t	1000	kg
minute	(′)	—	—				
second	(″)	—	—	Nucleonics, Radiation			
				becquerel	Bq	1.0	s^{-1}
Area				gray	Gy	1.0	J/kg
are	a	100	m^2	curie	Ci	3.7×10^{10}	Bq
hectare	ha	0.01	km^2	rad	rd	0.01	Gy
barn	barn	10^{-28}	m^2	roentgen	R	2.6×10^{-4}	C/kg
Energy				Pressure			
erg	erg	0.1	μJ	bar	b	100	kPa
calorie	cal	4.186	J	torr	Torr	133.3	Pa
electron-volt	eV	0.160	aJ	Time			
gauss-oersted	Ga Oe	7.96	μJ/m^3	minute	min	60	s
Force				hour	h	3600	s
dyne	dyn	10	μN	day	d	86 400	s
Length				Volume			
Ångstrom	Å	0.1	μm	litre	l or L	1.0	dm^3

Table 1.4 Conversion factors

Length [m]	
1 mil	25.40 μm
1 in	25.40 mm
1 ft	0.3048 m
1 yd	0.9144 m
1 fathom	1.829 m
1 mile	1.6093 km
1 nautical mile	1.852 km
Area [m^2]	
1 circular mil	506.7 μm^2
1 in^2	645.2 mm^2
1 ft^2	0.0929 m^2
1 yd^2	0.8361 m^2
1 acre	4047 m^2
1 $mile^2$	2.590 km^2
Volume [m^3]	
1 in^3	16.39 cm^3
1 ft^3	0.0283 m^3
1 yd^3	0.7646 m^3
1 UKgal	4.546 dm^3
Velocity [m/s, rad/s]	
Acceleration [m/s^2, rad/s^2]	
1 ft/min	5.080 mm/s
1 in/s	25.40 mm/s
1 ft/s	0.3048 m/s
1 mile/h	0.4470 m/s
1 knot	0.5144 m/s
1 deg/s	17.45 mrad/s
1 rev/min	0.1047 rad/s
1 rev/s	6.283 rad/s
1 ft/s^2	0.3048 m/s^2
1 mile/h per s	0.4470 m/s^2
Mass [kg]	
1 oz	28.35 g
1 lb	0.454 kg
1 slug	14.59 kg
1 cwt	50.80 kg
1 UKton	1016 kg
Energy [J], Power [W]	
1 ft lbf	1.356 J
1 m kgf	9.807 J
1 Btu	1055 J
1 therm	105.5 kJ
1 hp h	2.685 MJ
1 kW h	3.60 MJ
1 Btu/h	0.293 W
1 ft lbf/s	1.356 W
1 m kgf/s	9.807 W
1 hp	745.9 W
Thermal quantities [W, J, kg, K]	
1 W/in^2	1.550 kW/m^2
1 Btu/(ft^2 h)	3.155 W/m^2
1 Btu/(ft^3 h)	10.35 W/m^3
1 Btu/(ft h °F)	1.731 W/(m K)
1 ft lbf/lb	2.989 J/kg
1 Btu/lb	2326 J/kg
1 Btu/ft^3	37.26 KJ/m^3
1 ft lbf/(lb °F)	5.380 J/(kg K)
1 Btu/(lb °F)	4.187 kJ/(kg K)
1 Btu/(ft^3 °F)	67.07 kJ/m^3 K
Density [kg/m, kg/m^3]	
1 lb/in	17.86 kg/m
1 lb/ft	1.488 kg/m
1 lb/yd	0.496 kg/m
1 lb/in^3	27.68 Mg/m^3
1 lb/ft^3	16.02 kg/m^3
1 ton/yd^3	1329 kg/m^3
Flow rate [kg/s, m^3/s]	
1 lb/h	0.1260 g/s
1 ton/h	0.2822 kg/s
1 lb/s	0.4536 kg/s
1 ft^3/h	7.866 cm^3/s
1 ft^3/s	0.0283 m^3/s
1 gal/h	1.263 cm^3/s
1 gal/min	75.77 cm^3/s
1 gal/s	4.546 dm^3/s
Force [N], Pressure [Pa]	
1 dyn	10.0 μN
1 kgf	9.807 N
1 ozf	0.278 N
1 lbf	4.445 N
1 tonf	9.964 kN
1 dyn/cm^2	0.10 Pa
1 lbf/ft^2	47.88 Pa
1 lbf/in^2	6.895 kPa
1 $tonf/ft^2$	107.2 kPa
1 $tonf/in^2$	15.44 MPa
1 kgf/m^2	9.807 Pa
1 kgf/cm^2	98.07 kPa
1 mmHg	133.3 Pa
1 inHg	3.386 kPa
1 inH_2O	149.1 Pa
1 ftH_2O	2.989 kPa
Torque [N m]	
1 ozf in	7.062 nN m
1 lbf in	0.113 N m
1 lbf ft	1.356 N m
1 tonf ft	3.307 kN m
1 kgf m	9.806 N m
Inertia [kg m^2]	
Momentum [kg m/s, kg m^2/s]	
1 oz in^2	0.018 g m^2
1 lb in^2	0.293 g m^2
1 lb ft^2	0.0421 kg m^2
1 slug ft^2	1.355 kg m^2
1 ton ft^2	94.30 kg m^2
1 lb ft/s	0.138 kg m/s
1 lb ft^2/s	0.042 kg m^2/s
Viscosity [Pa s, m^2/s]	
1 poise	9.807 Pa s
1 kgf s/m^2	9.807 Pa s
1 lbf s/ft^2	47.88 Pa s
1 lbf h/ft^2	172.4 kPa s
1 stokes	1.0 cm^2/s
1 in^2/s	6.452 cm^2/s
1 ft^2/s	929.0 cm^2/s
Illumination [cd, lm]	
1 lm/ft^2	10.76 lm/m^2
1 cd/ft^2	10.76 cd/m^2
1 cd/in^2	1550 cd/m^2

Table 1.5 Relation between SI, e.s. and e.m. units

Quantity	*SI unit*	*Equivalent number n of*			
		e.s.u.		*e.m.u.*	
Length	m	10^2	cm	10^2	cm
Mass	kg	10^3	g	10^3	g
Time	s	1	s	1	s
Force	N	10^5	dyn	10^5	dyn
Torque	N m	10^7	dyn cm	10^7	dyn cm
Energy	J	10^7	erg	10^7	erg
Power	W	10^7	erg/s	10^7	erg/s
Charge, electric flux	C	3×10^9	stC	10^{-1}	abC
density	C/m^2	3×10^5	stC/cm^2	10^{-5}	abC/cm^2
Potential, e.m.f.	V	$10^{-2}/3$	stV	10^8	abV
Electric field strength	V/m	$10^{-4}/3$	stV/cm	10^6	abV/cm
Current	A	3×10^9	stA	10^{-1}	abA
density	A/m^2	3×10^5	stA/cm^2	10^{-5}	abA/cm^2
Magnetic flux	Wb	$10^{-2}/3$	stWb	10^8	Mx
density	T	$10^{-6}/3$	$stWb/cm^2$	10^4	Gs
Mag. fd. strength	A/m	$12\pi \times 10^7$	stA/cm	$4\pi \times 10^{-3}$	Oe
M.M.F.	A	$12\pi \times 10^9$	stA	$4\pi \times 10^{-1}$	Gb
Resistivity	Ω m	$10^{-9}/9$	stΩ cm	10^{11}	abΩ cm
Conductivity	S/m	9×10^9	stS/cm	10^{-11}	abS/cm
Permeability (abs)	H/m	$10^{-13}/36\pi ς$	—	$10^7/4\pi ς$	—
Permittivity (abs)	F/m	$36\pi \times 10^9$	—	$4\pi \times 10^{-11}$	—
Resistance	Ω	$10^{-11}/9$	stΩ	10^9	abΩ
Conductance	S	9×10^{11}	stS	10^{-9}	abS
Inductance	H	$10^{-12}/9$	stH	10^9	cm
Capacitance	F	9×10^{11}	cm	9×10^{11}	abF
Reluctance	A/Wb	$36\pi \times 10^{11}$	—	$4\pi \times 10^{-8}$	Gb/Mx
Permeance	Wb/A	$10^{11}/36\pi ς$	—	$10^9/4\pi ς$	Mx/Gb

Gb = gilbert; Gs = gauss; Mx = maxwell; Oe = oersted.

1.2.1 Trigonometric relations

The trigonometric functions (sine, cosine, tangent, cosecant, secant, cotangent) of an angle θ are based on the circle, given by $x^2 + y^2 = h^2$. Let two radii of the circle enclose an angle θς and form the sector area $S_c = (\pi h^2)(\theta/2\pi)$ shown shaded in *Figure 1.1* (left): then θς can be defined as $2S_c/h^2$. The *right-angled* triangle with sides h (hypotenuse), a (adjacent side) and p (opposite side) give ratios defining the trigonometric functions

$$\sin\theta = p/h \qquad \text{cosec}\ \theta = 1/\sin\theta = h/p$$
$$\cos\theta = a/h \qquad \sec\ \theta = 1/\cos\theta = h/a$$
$$\tan\theta = p/a \qquad \text{cotan}\ \theta = 1/\tan\theta = a/p$$

In *any* triangle (*Figure 1.1*, right) with angles, A, B and C at the corners opposite, respectively, to sides a, b and c, then $A + B + C = \pi$ rad (180°) and the following relations hold:

$$a = b\cos C + c\cos B$$
$$b = c\cos A + a\cos C$$
$$c = a\cos B + b\cos A$$
$$a/\sin A = b/\sin B = c/\sin C$$
$$a^2 = b^2 + c^2 + 2bc\cos A$$
$$(a+b)/(a-b) = (\sin A + \sin B)/(\sin A - \sin B) \Leftarrow$$

Other useful relationships are:

$$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$$
$$\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$$
$$\tan(x \pm y) = (\tan x \cdot \tan y)/(1 \mp \tan x \cdot \tan y) \Leftarrow$$
$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \cos^2 x = -\tfrac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x + \cos^2 x = 1 \qquad \sin^3 x = -\tfrac{1}{4}(3\sin x - \sin 3x)$$
$$\cos^3 x = \tfrac{1}{4}(3\cos x + \cos 3x)$$

$$\sin x \pm \sin y = 2\left[\begin{matrix}\cos\\ \sin\end{matrix}\tfrac{1}{2}(x-y) \cdot \begin{matrix}\sin\\ \Leftarrow\\ \cos\end{matrix}\tfrac{1}{2}(x+y)\right]\Bigg]$$

$$\cos x \pm \cos y = -2\left[\begin{matrix}\cos\\ \sin\end{matrix}\tfrac{1}{2}(x-y) \cdot \begin{matrix}\sin\\ \Leftarrow\\ \cos\end{matrix}\tfrac{1}{2}(x+y)\right]\Bigg]$$

$$\tan x \pm \tan y = \sin(x \pm y)/\cos x \cdot \cos y$$
$$\sin^2 x - \sin^2 y = \sin(x+y) \cdot \sin(x-y) \Leftarrow$$
$$\cos^2 x - \cos^2 y = -\sin(x+y) \cdot \sin(x-y) \Leftarrow$$
$$\cos^2 x - \sin^2 y = \cos(x+y) \cdot \cos(x-y) \Leftarrow$$

$$d(\sin x)/dx = \cos x \qquad \int \sin x \cdot dx = -\cos x + k$$
$$d(\cos x)/dx = -\sin x \qquad \int \cos x \cdot dx = \sin x + k$$
$$d(\tan x)/dx = \sec^2 x \qquad \int \tan x \cdot dx = -\ln|\cos x| + k$$

Values of sin θ, cos θ and tan θ for $0° \Leftarrow < \theta < 90°$ (or $0 < \theta$ ς < 1.571 rad) are given in *Table 1.7* as a check list, as they can generally be obtained directly from calculators.

Table 1.6 Mathematical symbolism

Term	*Symbol*
Base of natural logarithms	e (= 2.718 28 . . .)
Complex number	$\mathbf{C} = A + jB = C\exp(j\theta)$ $= C \angle \theta$
argument; modulus	arg $\mathbf{C} = \theta$; mod $\mathbf{C} = C$
conjugate	$\mathbf{C}^* = A - jB = C\exp(-j\theta)$ $= C \angle -\theta$
real part; imaginary part	Re $\mathbf{C} = A$; Im $\mathbf{C} = B$
Co-ordinates	
cartesian	x, y, z
cylindrical; spherical	r, ϕ, z; r, θ, ϕ
Function of x	
general	$f(x)$, $g(x)$, $F(x)$
Bessel	$J_n(x)$
circular	$\sin x$, $\cos x$, $\tan x \ldots$
inverse	$\arcsin x$, $\arccos x$, $\arctan x \ldots$
differential	dx
partial	∂x
exponential	$\exp(x)$
hyperbolic	$\sinh x$, $\cosh x$, $\tanh x \ldots$
inverse	$\mathrm{arsinh}\, x$, $\mathrm{arcosh}\, x$, $\mathrm{artanh}\, x \ldots$
increment	Δx, δx
limit	$\lim x$
logarithm	
base b	$\log_b x$
common; natural	$\lg x$; $\ln x$ (or $\log x$; $\log_e x$)
Matrix	A, B
complex conjugate	A^*, B^*
product	AB
square, determinant	$\det A$
inverse	A^{-1}
transpose	A^t
unit	I
Operator	
Heaviside	$p\ (\equiv d/dt)$
impulse function	$\delta(t)$
Laplace $L[f(t)] = F(s)$	$s\ (= \sigma + j\omega)$
nabla, del	∇
rotation $\pi/2$ rad;	j
$2\pi/3$ rad	h
step function	$H(t)$, $u(t)$
Vector	**A, a, B, b**
curl of A	curl $\mathbf{A}$, $\nabla \times \mathbf{A}$
divergence of A	div $\mathbf{A}$, $\nabla \cdot \mathbf{A}$
gradient of ϕ	grad ϕ, $\nabla\phi$
product: scalar; vector	$\mathbf{A} \cdot \mathbf{B}$; $\mathbf{A} \times \mathbf{B}$
units in cartesian axes	**i, j, k**

Table 1.7 Trigonometric functions of θ

θ deg	θ rad	*sin* θ	*cos* θ	*tan* θ
0	0.0	0.0	1.0	0.0
5	0.087	0.087	0.996	0.087
10	0.175	0.174	0.985	0.176
15	0.262	0.259	0.966	0.268
20	0.349	0.342	0.940	0.364
25	0.436	0.423	0.906	0.466
30	0.524	0.500	0.866	0.577
35	0.611	0.574	0.819	0.700
40	0.698	0.643	0.766	0.839
45	0.766	0.707	0.707	1.0
50	0.873	0.766	0.643	1.192
55	0.960	0.819	0.574	1.428
60	1.047	0.866	0.500	1.732
65	1.134	0.906	0.423	2.145
70	1.222	0.940	0.342	2.747
75	1.309	0.966	0.259	3.732
80	1.396	0.985	0.174	5.671
85	1.484	0.996	0.097	11.43
90	1.571	1.0	0.0	∞

1.2.2 Exponential and hyperbolic relations

Exponential functions For a positive datum ('real') number u, the exponential functions $\exp(u)$ and $\exp(-u)$ are given by the summation to infinity of the series

$$\exp(\pm u) = 1 \pm u + u^2/2! \pm u^3/3! + u^4/4! \pm \cdots$$

with $\exp(+u)$ increasing and $\exp(-u)$ decreasing at a rate proportional to u.

If $u = 1$, then

$$\exp(+1) = 1 + 1 + 1/2 + 1/6 + 1/24 + \cdots = e = 2.718 \cdots$$
$$\exp(-1) = 1 - 1 + 1/2 - 1/6 + 1/24 - \cdots = 1/e = 0.368 \cdots$$

In the electrical technology of transients, u is most commonly a negative function of time t given by $u = -(t/T)$. It then has the graphical form shown in *Figure 1.2* (left) as a time dependent variable. With an initial value k, i.e. $y = k\exp(-t/T)$, the rate of reduction with time is $dy/dt = -(k/T)\exp(-t/T)$. The initial rate at $t = 0$ is $-k/T$. If this rate were maintained, y would reach zero at $t = T$, defining the *time constant* T. Actually, after time T the value of y is $k\exp(-t/T) = k\exp(-1) = 0.368k$. Each successive interval T decreases y by the factor 0.368. At a time $t = 4.6T$ the value of y is $0.01k$, and at $t = 6.9T$ it is $0.001k$.

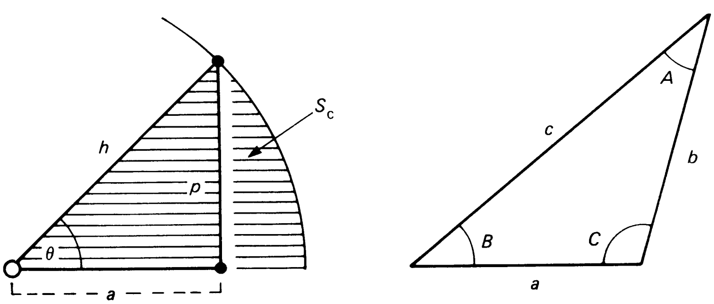


Figure 1.1 Trigonometric relations

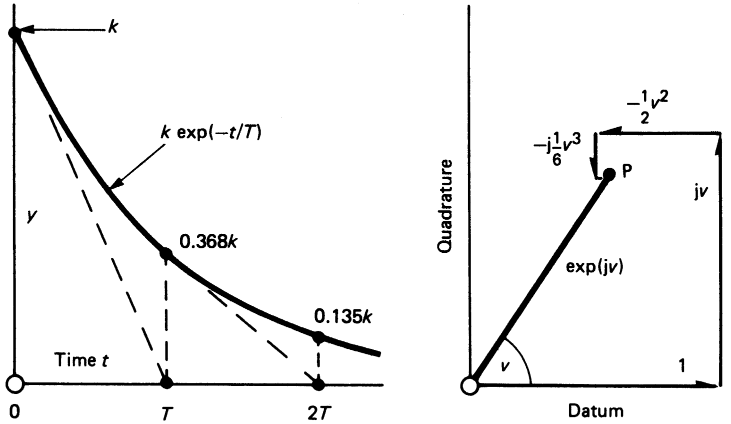


Figure 1.2 Exponential relations

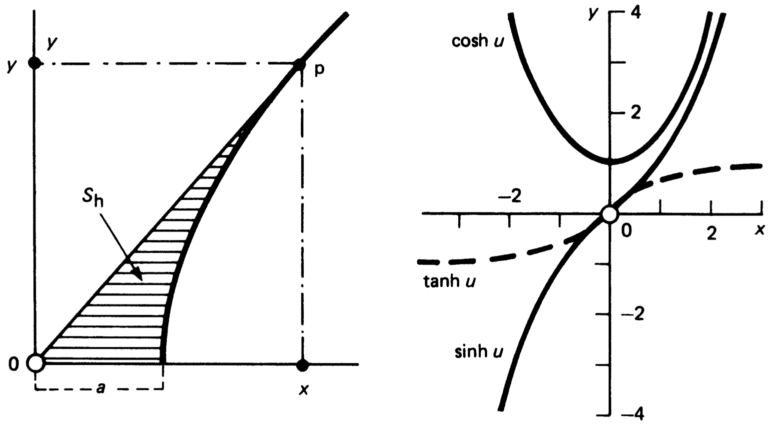


Figure 1.3 Hyperbolic relations

If u is a quadrature ('imaginary') number $\pm jv$, then

$$\exp(\pm jv) = 1 \pm jv - v^2/2! \mp jv^3/3! + v^4/4! \pm \ldots$$

because $j^2 = -1$, $j^3 = -j1$, $j^4 = +1$, etc. *Figure 1.2* (right) shows the summation of the first five terms for exp(j1), i.e.

$$\exp(j1) = 1 + j1 - 1/2 - j1/6 + 1/24$$

a complex or expression converging to a point P. The length OP is unity and the angle of OP to the datum axis is, in fact, 1 rad. In general, exp(jv) is equivalent to a shift by $\angle v$ rad. It follows that $\exp(\pm jv) = \cos v \pm j \sin v$, and that

$$\exp(jv) + \exp(-jv) = 2\cos v \quad \exp(jv) - \exp(-jv) = j2\sin v$$

For a complex number $(u + jv)$, then

$$\exp(u + jv) = \exp(u) \cdot \exp(jv) = \exp(u) \cdot \angle v$$

Hyperbolic functions A point P on a rectangular hyperbola $(x/a)^2 - (y/a)^2 = 1$ defines the hyperbolic 'sector' area $S_h = \frac{1}{2}a^2 \ln[(x/a - (y/a)]$ shown shaded in *Figure 1.3* (left). By analogy with $\theta = 2S_c/h^2$ for the trigonometrical angle θ, the hyperbolic entity (not an angle in the ordinary sense) is $u = 2S_h/a^2$, where a is the major semi-axis. Then the hyperbolic functions of u for point P are:

$\sinh u = y/a$	$\text{cosech } u = a/y$
$\cosh u = x/a$	$\text{sech } u = a/x$
$\tanh u = y/x$	$\coth u = x/y$

The principal relations yield the curves shown in the diagram (right) for values of u between 0 and 3. For higher values sinh u approaches ±cosh u, and tanh u becomes asymptotic to ±1. Inspection shows that $\cosh(-u) = \cosh u$, $\sinh(-u) = -\sinh u$ and $\cosh^2 u - \sinh^2 u = 1$.

The hyperbolic functions can also be expressed in the exponential form through the series

$$\cosh u = 1 + u^2/2! + u^4/4! + u^6/6! + \cdots$$
$$\sinh u = u + u^3/3! + u^5/5! + u^7/7! + \cdots$$

so that

$$\cosh u = \tfrac{1}{2}[\exp(u) + \exp(-u)] \quad \sinh u = \tfrac{1}{2}[\exp(u) - \exp(-u)]$$
$$\cosh u + \sinh u = \exp(u) \qquad \cosh u - \sinh u = \exp(-u)$$

Other relations are:

$$\sinh u + \sinh v = 2\sinh\tfrac{1}{2}(u + v) \cdot \cosh\tfrac{1}{2}(u - v)$$
$$\cosh u + \cosh v = 2\cosh\tfrac{1}{2}(u + v) \cdot \cosh\tfrac{1}{2}(u - v)$$
$$\cosh u - \cosh v = 2\sinh\tfrac{1}{2}(u + v) \cdot \sinh\tfrac{1}{2}(u - v)$$
$$\sinh(u \pm v) = \sinh u \cdot \cosh v \pm \cosh u \cdot \sinh v$$
$$\cosh(u \pm v) = \cosh u \cdot \cosh v \pm \sinh u \cdot \sinh v$$
$$\tanh(u \pm v) = (\tanh u \pm \tanh v)/(1 \pm \tanh u \cdot \tanh v)$$

Table 1.8 Exponential and hyperbolic functions

u	*exp*(*u*)	*exp*(−*u*)	*sinh u*	*cosh u*	*tanh u*
0.0	1.0	1.0	0.0	1.0	0.0
0.1	1.1052	0.9048	0.1092	1.0050	0.0997
0.2	1.2214	0.8187	0.2013	1.0201	0.1974
0.3	1.3499	0.7408	0.3045	1.0453	0.2913
0.4	1.4918	0.6703	0.4108	1.0811	0.3799
0.5	1.6487	0.6065	0.5211	1.1276	0.4621
0.6	1.8221	0.5488	0.6367	1.1855	0.5370
0.7	2.0138	0.4966	0.7586	1.2552	0.6044
0.8	2.2255	0.4493	0.8881	1.3374	0.6640
0.9	2.4596	0.4066	1.0265	1.4331	0.7163
1.0	2.7183	0.3679	1.1752	1.5431	0.7616
1.2	3.320	0.3012	1.5095	1.8107	0.8337
1.4	4.055	0.2466	1.9043	2.1509	0.8854
1.6	4.953	0.2019	2.376	2.577	0.9217
1.8	6.050	0.1653	2.942	3.107	0.9468
2.0	7.389	0.1353	3.627	3.762	0.9640
2.303	10.00	0.100	4.950	5.049	0.9802
2.5	12.18	0.0821	6.050	6.132	0.9866
2.75	15.64	0.0639	7.789	7.853	0.9919
3.0	20.09	0.0498	10.02	10.07	0.9951
3.5	33.12	0.0302	16.54	16.57	0.9982
4.0	54.60	0.0183	27.29	27.31	0.9993
4.5	90.02	0.0111	45.00	45.01	0.9998
4.605	100.0	0.0100	49.77	49.80	0.9999
5.0	148.4	0.0067	74.20	74.21	0.9999
5.5	244.7	0.0041	122.3	cosh *u*	tanh *u*
6.0	403.4	0.0025	201.7	= sinh *u*	= 1.0
6.908	1000	0.0010	500	$= \frac{1}{2}\exp(u)$	

$$\sinh(u \pm jv) = (\sinh u \cdot \cos v) \pm j(\cosh u \cdot \sin v)$$

$$\cosh(u \pm jv) = (\cosh u \cdot \cos v) \pm j(\sinh u \cdot \sin v)$$

$$d(\sinh u)/du = \cosh u \qquad \int \sinh u \cdot du = \cosh u$$

$$d(\cosh u)/du = \sinh u \qquad \int \cosh u \cdot du = \sinh u$$

Exponential and hyperbolic functions of u between zero and 6.908 are listed in *Table 1.8*. Many calculators can give such values directly.

1.2.3 Bessel functions

Problems in a wide range of technology (e.g. in eddy currents, frequency modulation, etc.) can be set in the form of the Bessel equation

$$\frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} + \left[1 - \frac{n^2}{x^2}\right] y = 0$$

and its solutions are called Bessel functions of order n. For $n = 0$ the solution is

$$J_0(x) = 1 - (x^2/2^2) + (x^4/2^2 \cdot 4^2) - (x^6/2^2 \cdot 4^2 \cdot 6^2) + \cdots$$

and for $n = 1, 2, 3 \ldots$

$$J_n(x) = \frac{x^n}{2^n n!}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots\right]$$

Table 1.9 gives values of $J_n(x)$ for various values of n and x.

1.2.4 Series

Factorials In several of the following the factorial (n!) of integral numbers appears. For n between 2 and 10 these are

2! = 2	1/2! = 0.5
3! = 6	1/3! = 0.1667
4! = 24	$1/4! = 0.417 \times 10^{-1}$
5! = 120	$1/5! = 0.833 \times 10^{-2}$
6! = 720	$1/6! = 0.139 \times 10^{-2}$
7! = 5 040	$1/7! = 0.198 \times 10^{-3}$
8! = 40 320	$1/8! = 0.248 \times 10^{-4}$
9! = 362 880	$1/9! = 0.276 \times 10^{-5}$
10! = 3 628 800	$1/10! = 0.276 \times 10^{-6}$

Progression

Arithmetic $a + (a+d) + (a+2d) + \cdots + [a + (n-1)d]$
$= \frac{1}{2} n$ (sum of 1st and nth terms)

Geometric $a + ar + ar^2 + \cdots + ar^{n-1} = a(1-r^n)/(1-r)$

Trigonometric See Section 1.2.1.

Exponential and hyperbolic See Section 1.2.2.

Binomial

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + (-1)^r \frac{n!}{r!(n-r)!}x^r + \cdots$$

$$(a \pm x)^n = a^n[1 \pm (x/a)]^n$$

Binomial coefficients $n!/[r!\,(n-r)!]$ are tabulated:

Term $r=$	0	1	2	3	4	5	6	7	8	9	10
$n=1$	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Power If there is a power series for a function $f(h)$, it is given by

$$f(h) = f(0) + hf^{(i)}(0) + (h^2/2!)f^{(ii)}(0) + (h^3/3!)f^{(iii)}(0) + \cdots + (h^r/r!)f^{(r)}(0) + \cdots \qquad (\textit{Maclaurin})$$

$$f(x+h) = f(x) + hf^{(i)}(x) + (h^2/2!)f^{(ii)}(x) + \cdots + (h^r/r!)f^{(r)}(x) + \cdots \qquad (\textit{Taylor})$$

Permutation, combination

$$^nP_r = n(n-1)(n-2)(n-3)\ldots(n-r+1) = n!/(n-r)!$$

$$^nC_r = (1/r!)[n(n-1)(n-2)(n-3)\ldots(n-r+1)] = n!/r!(n-r)!$$

Bessel See Section 1.2.3.
Fourier See Section 1.2.5.

1.2.5 Fourier series

A univalued periodic wave form $f(\theta)$ of period 2π is represented by a summation in general of sine and cosine waves of fundamental period 2π and of integral harmonic orders n $(=2, 3, 4, \ldots)$ as

$$f(\theta) = c_0 + a_1\cos\theta + a_2\cos 2\theta + \cdots + a_n\cos n\theta + \cdots + b_1\sin\theta + b_2\sin 2\theta + \cdots + b_n\sin n\theta + \cdots$$

The mean value of $f(\theta)$ over a full period 2π is

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cdot \mathrm{d}\theta$$

and the harmonic-component amplitudes a and b are

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cdot\cos n\theta\,\mathrm{d}\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cdot\sin n\theta\,\mathrm{d}\theta$$

Table 1.10 gives for a number of typical wave forms the harmonic series in square brackets, preceded by the mean value c_0 where it is not zero.

1.2.6 Derivatives and integrals

Some basic forms are listed in *Table 1.11*. Entries in a given column are the integrals of those in the column to its left and the derivatives of those to its right. Constants of integration are omitted.

1.2.7 Laplace transforms

Laplace transformation is a method of deriving the response of a system to any stimulus. The system has a basic equation of behaviour, and the stimulus is a pulse, step, sine wave or other variable with time. Such a response involves integration: the Laplace transform method removes integration difficulties, as tables are available for the direct solution of a great variety of problems. The process is analogous to evaluation (for example) of $y = 2.1^{3.6}$ by *transformation* into a logarithmic form $\log y = 3.6 \times \log(2.1)$, and a subsequent *inverse transformation* back into arithmetic by use of a table of antilogarithms.

The Laplace transform (L.t.) of a time-varying function $f(t)$ is

$$\mathrm{L}[f(t)] = \mathrm{F(s)} = \int_0^{\infty} \exp(-st)\cdot f(t)\cdot \mathrm{d}t$$

and the inverse transformation of F(s) to give $f(t)$ is

$$\mathrm{L}^{-1}[\mathrm{F(s)}] = f(t) = \lim \frac{1}{2\pi}\int_{c-\mathrm{j}\omega}^{c+\mathrm{j}\omega} \exp(st)\cdot \mathrm{F(s)}\cdot \mathrm{ds}$$

The process, illustrated by the response of a current $i(t)$ in an electrical network of impedance z to a voltage $v(t)$ applied at $t=0$, is to write down the transform equation

$$\mathrm{I(s)} = \mathrm{V(s)/Z(s)}$$

where I(s) is the L.t. of the current $i(t)$, V(s) is the L.t. of the voltage $v(t)$, and Z(s) is the *operational* impedance. Z(s) is obtained from the network resistance R, inductance L and capacitance C by leaving R unchanged but replacing L by Ls and C by $1/C$s. The process is equivalent to writing the network impedance for a steady state frequency ω and then replacing jω by s. V(s) and Z(s) are polynomials in s: the quotient V(s)/Z(s) is reduced algebraically to a form recognisable in the transform table. The resulting current/time relation $i(t)$ is read out: it contains the complete solution. However, if at $t=0$ the network has initial energy (i.e. if currents flow in inductors or charges are stored in capacitors), the equation becomes

$$\mathrm{I(s)} = [\mathrm{V(s)} + \mathrm{U(s)}]/\mathrm{Z(s)}$$

where U(s) contains such terms as LI_0 and $(1/\mathrm{s})V_0$ for the inductors or capacitors at $t=0$.

A number of useful transform pairs is listed in *Table 1.12*.

1.2.8 Binary numeration

A number N in decimal notation can be represented by an ordered set of binary digits $a_n, a_{n-2}, \ldots, a_2, a_1, a_0$ such that

$$N = 2^n a_n + 2^{n-1}a_{n-1} + \cdots + 2a_1 + a_0$$

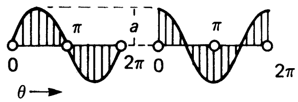
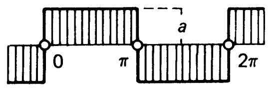
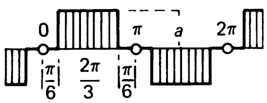
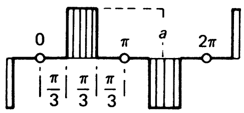
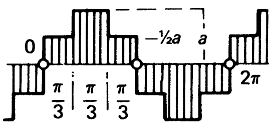
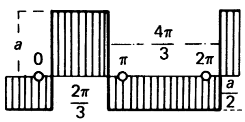
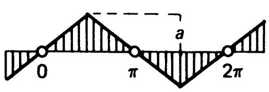
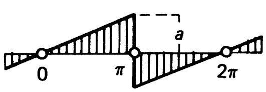
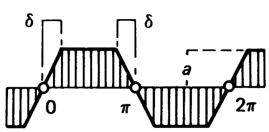
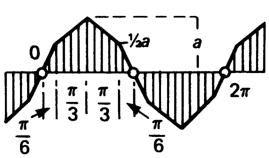
Decimal	1	2	3	4	5	6	7	8	9	10	100
Binary	1	10	11	100	101	110	111	1000	1001	1010	1100100

Table 1.9 Bessel functions $J_n(x)$

n	$J_n(1)$	$J_n(2)$	$J_n(3)$	$J_n(4)$	$J_n(5)$	$J_n(6)$	$J_n(7)$	$J_n(8)$	$J_n(9)$	$J_n(10)$	$J_n(11)$	$J_n(12)$	$J_n(13)$	$J_n(14)$	$J_n(15)$
0	0.7652	0.2239	−0.2601	−0.3971	−0.1776	0.1506	0.3001	0.1717	−0.0903	−0.2459	−0.1712	0.0477	0.2069	0.1711	−0.0142
1	0.4401	0.5767	0.3391	−0.0660	−0.3276	−0.2767	−0.0047	0.2346	0.2453	0.0435	−0.1768	−0.2234	−0.0703	0.1334	0.2051
2	0.1149	0.3528	0.4861	0.3641	0.0466	−0.2429	−0.3014	−0.1130	0.1448	0.2546	0.1390	−0.0849	−0.2177	−0.1520	0.0416
3	0.0196	0.1289	0.3091	0.4302	0.3648	0.1148	−0.1676	−0.2911	−0.1809	0.0584	0.2273	0.1951	0.0033	−0.1768	−0.1940
4	—	0.0340	0.1320	0.2811	0.3912	0.3567	0.1578	−0.1054	−0.2655	−0.2196	−0.0150	0.1825	0.2193	0.0762	−0.1192
5	—	—	0.0430	0.1321	0.2611	0.3621	0.3479	0.1858	−0.0550	−0.2341	−0.2383	−0.0735	0.1316	0.2204	0.1305
6	—	—	0.0114	0.0491	0.1310	0.2458	0.3392	0.3376	0.2043	−0.0145	−0.2016	−0.2437	−0.1180	0.0812	0.2061
7	—	—	—	0.0152	0.0534	0.1296	0.2336	0.3206	0.3275	0.2167	0.0184	−0.1703	−0.2406	−0.1508	0.0345
8	—	—	—	—	0.0184	0.0565	0.1280	0.2235	0.3051	0.3179	0.2250	0.0451	−0.1410	−0.2320	−0.1740
9	—	—	—	—	—	0.0212	0.0589	0.1263	0.2149	0.2919	0.3089	0.2304	0.0670	−0.1143	−0.2200
10	—	—	—	—	—	—	0.0235	0.0608	0.1247	0.2075	0.2804	0.3005	0.2338	0.0850	−0.0901
11	—	—	—	—	—	—	—	0.0256	0.0622	0.1231	0.2010	0.2704	0.2927	0.2357	0.0999
12	—	—	—	—	—	—	—	—	0.0274	0.0634	0.1216	0.1953	0.2615	0.2855	0.2367
13	—	—	—	—	—	—	—	—	0.0108	0.0290	0.0643	0.1201	0.1901	0.2536	0.2787
14	—	—	—	—	—	—	—	—	—	0.0119	0.0304	0.0650	0.1188	0.1855	0.2464
15	—	—	—	—	—	—	—	—	—	—	0.0130	0.0316	0.0656	0.1174	0.1813

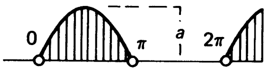
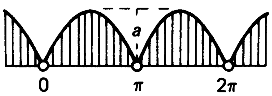
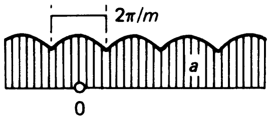
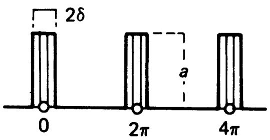
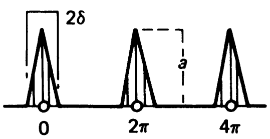
Values below 0.01 not tabulated.

Table 1.10 Fourier series

	Wave form	*Series*
1		Sine: $a \sin \theta$ Cosine: $a \sin \theta$
2		Square: $a\frac{4}{\pi}\left[\frac{\sin\theta}{1}+\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+\frac{\sin 7\theta}{7}+\cdots\right]$
5		Rectangular block: $a\frac{2\sqrt{3}}{\pi}\left[\frac{\sin\theta}{1}-\frac{\sin 5\theta}{5}-\frac{\sin 7\theta}{7}+\frac{\sin 11\theta}{11}+\frac{\sin 13\theta}{13}-\frac{\sin 17\theta}{17}-\cdots\right]$
4		Rectangular block: $a\frac{4}{\pi}\left[\frac{\sin\theta}{2\cdot 1}-\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{2\cdot 5}+\frac{\sin 7\theta}{2\cdot 7}-\frac{\sin 9\theta}{9}+\frac{\sin 11\theta}{2\cdot 11}+\frac{\sin 13\theta}{2\cdot 13}-\frac{\sin 15\theta}{15}+\frac{\sin 17\theta}{2\cdot 17}+\cdots\right]$
4a		Stepped rectangle: $a\frac{3}{\pi}\left[\frac{\sin\theta}{1}+\frac{\sin 5\theta}{5}+\frac{\sin 7\theta}{7}+\frac{\sin 11\theta}{11}+\frac{\sin 13\theta}{13}+\frac{\sin 17\theta}{17}+\cdots\right]$
3		Asymmetric rectangle: $a\frac{3\sqrt{3}}{2\pi}\left[\frac{\sin\theta}{1}-\frac{\sin 5\theta}{5}-\frac{\sin 7\theta}{7}+\frac{\sin 11\theta}{11}+\frac{\sin 13\theta}{13}-\cdots-\frac{\cos 2\theta}{2}-\frac{\cos 4\theta}{4}+\frac{\cos 8\theta}{8}+\frac{\cos 10\theta}{10}-\cdots\right]$
		Triangle: $a\frac{8}{\pi^2}\left[\frac{\sin\theta}{1}-\frac{\sin 3\theta}{9}+\frac{\sin 5\theta}{25}-\frac{\sin 7\theta}{49}+\frac{\sin 9\theta}{81}-\frac{\sin 11\theta}{121}+\cdots\right]$
		Sawtooth: $a\frac{3}{\pi}\left[\frac{\sin\theta}{1}-\frac{\sin 2\theta}{2}+\frac{\sin 3\theta}{3}-\frac{\sin 4\theta}{4}+\frac{\sin 5\theta}{5}-\cdots\right]$
		Trapeze: $a\frac{4}{\pi\delta}\left[\frac{\sin\delta\sin\theta}{1}+\frac{\sin 3\delta\sin 3\theta}{9}+\frac{\sin 5\delta\sin 5\theta}{25}+\cdots\right]$ $a\frac{6\sqrt{3}}{\pi^2}\left[\frac{\sin\theta}{1}-\frac{\sin 5\theta}{25}+\frac{\sin 7\theta}{49}-\frac{\sin 11\theta}{121}+\cdots\right]$ for $\delta=\pi/3$
		Trapeze-triangle: $a\frac{9}{\pi^2}\left[\frac{\sin\theta}{1}+\frac{\sin 5\theta}{25}-\frac{\sin 7\theta}{49}+\frac{\sin 11\theta}{121}-\frac{\sin 13\theta}{169}+\cdots\right]$

cont'd

Table 1.10 (*continued*)

Wave form	*Series*
	Rectified sine (half-wave): $a\frac{1}{\pi} + a\frac{2}{\pi}\left[\frac{\pi \sin\theta}{4} - \frac{\cos 2\theta}{1\cdot 3} - \frac{\cos 4\theta}{3\cdot 5} - \frac{\cos 6\theta}{5\cdot 7} - \cdots\right]$
	Rectified sine (full-wave): $a\frac{2}{\pi} - a\frac{4}{\pi}\left[\frac{\cos 2\theta}{1\cdot 3} + \frac{\cos 4\theta}{3\cdot 5} + \frac{\cos 6\theta}{5\cdot 7} + \frac{\cos 8\theta}{7\cdot 9} + \cdots\right]$
	Rectified sine (m-phase): $a\frac{m}{\pi}\sin\frac{\pi}{m} + a\frac{2m}{\pi}\sin\frac{\pi}{m}\left[\frac{\cos m\theta}{m^2-1} - \frac{\cos 2m\theta}{4m^2-1} + \frac{\cos 3m\theta}{9m^2-1} - \cdots\right]$
	Rectangular pulse train: $a\frac{\delta}{\pi} + a\frac{2}{\pi}\left[\frac{\sin\delta\cos\theta}{1} + \frac{\sin 2\delta\cos 2\theta}{2} + \frac{\sin 3\delta\cos 3\theta}{3} + \cdots\right]$ $a\frac{\delta}{\pi} + a\frac{2\delta}{\pi}\left[\frac{\cos\theta}{1} + \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} + \cdots\right]$ for $\delta \ll \pi$
	Triangular pulse train: $a\frac{\delta}{2\pi} + a\frac{4}{\pi\delta}\left[\frac{\sin^2(\frac{1}{2}\delta)}{1}\cos\theta + \frac{\sin^2 2(\frac{1}{2}\delta)}{4}\cos 2\theta + \frac{\sin^2 3(\frac{1}{2}\delta)}{9}\cos 3\theta + \cdots\right]$ $a\frac{\delta}{2\pi} + a\frac{\delta}{\pi}[\cos\theta + \cos 2\theta + \cos 3\theta + \cdots]$ for $\delta \ll \pi$

where the as have the values either 1 or 0. Thus, if $N = 19$,

$$19 = 16 + 2 + 1 = (2^4)1 + (2^3)0 + (2^2)0 + (2^1)1 + (2^0)1 = 10011$$

in binary notation. The rules of addition and multiplication are

$$0+0=0,\ 0+1=1,\ 1+1=10;\ 0\times 0=0,\ 0\times 1=0,\ 1\times 1=1$$

1.2.9 Power ratio

In communication networks the powers P_1 and P_2 at two specified points may differ widely as the result of amplification or attenuation. The power ratio P_1/P_2 is more convenient in logarithmic terms.

Neper [Np] This is the natural logarithm of a voltage or current ratio, given by

$$a = \ln(V_1/V_2) \quad \text{or} \quad a = \ln(I_1/I_2)\ \text{Np}$$

If the voltages are applied to, or the currents flow in, identical impedances, then the power ratio is

$$a = \ln(V_1/V_2)^2 = 2\ln(V_1/V_2)$$

and similarly for current.

Decibel [dB] The power gain is given by the common logarithm $\lg(P_1/P_2)$ in bel [B], or most commonly by $A = 10\log(P_1/P_2)$ decibel [dB]. With again the proviso that the powers are developed in identical impedances, the power gain is

$$A = 10\log(P_1/P_2) = 10\log(V_1/V_2)^2 = 20\log(V_1/V_2)\ \text{dB}$$

Table 1.13 gives the power ratio corresponding to a gain A (in dB) and the related identical-impedance voltage (or current) ratios. Approximately, 3 dB corresponds to a power ratio of 2, and 6 dB to a power ratio of 4. The decibel equivalent of 1 Np is 8.69 dB.

1.2.10 Matrices and vectors

1.2.10.1 Definitions

If $a_{11}, a_{12}, a_{13}, a_{14} \ldots$ is a set of elements, then the rectangular array

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \ldots a_{2n} \\ & & & \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} \ldots a_{mn} \end{bmatrix}$$

arranged in m rows and n columns is called an $(m \times n)$ *matrix*. If $m = n$ then $\mathbf{A}$ is *n-square*.

Table 1.11 Derivatives and integrals

$\mathrm{d}[f(x)]/\mathrm{d}x$	$f(x)$	$\int f(x)\cdot \mathrm{d}x$
1	x	$\frac{1}{2}x^2$
nx^{n-1}	$x^n\ (n \neq -1)$	$x^{n+1}/(n+1)$
$-1/x^2$	$1/x$	$\ln x$
$1/x$	$\ln x$	$x \ln x - x$
$\exp x$	$\exp x$	$\exp x$
$\cos x$	$\sin x$	$-\cos x$
$-\sin x$	$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$	$\ln(\sec x)$
$-\mathrm{cosec}\ x \cdot \cot x$	$\mathrm{cosec}\ x$	$\ln(\tan \frac{1}{2}x)$
$\sec x \cdot \tan x$	$\sec x$	$\ln(\sec x + \tan x)$
$-\mathrm{cosec}^2 x$	$\cot x$	$\ln(\sin x)$
$1/\sqrt{(a^2-x^2)}$	$\arcsin(x/a)$	$x \arcsin(x/a) + \sqrt{(a^2-x^2)}$
$-1/\sqrt{(a^2-x^2)}$	$\arccos(x/a)$	$x \arccos(x/a) - \sqrt{(a^2-x^2)}$
$a/(a^2+x^2)$	$\arctan(x/a)$	$x \arctan(x/a) - \frac{1}{2}a \ln(a^2+x^2)$
$-a/x\sqrt{(x^2-a^2)}$	$\mathrm{arccosec}(x/a)$	$x\ \mathrm{arccosec}(x/a) + a \ln \mid x + \sqrt{(x^2-a^2)} \mid$
$a/x\sqrt{(x^2-a^2)}$	$\mathrm{arcsec}(x/a)$	$x\ \mathrm{arcsec}(x/a) - a \ln \mid x + \sqrt{(x^2-a^2)} \mid$
$-a/(a^2+x^2)$	$\mathrm{arccot}(x/a)$	$x\ \mathrm{arccot}(x/a) + \frac{1}{2}a \ln(a^2+x^2)$
$\cosh x$	$\sinh x$	$\cosh x$
$\sinh x$	$\cosh x$	$\sinh x$
$\mathrm{sech}^2 x$	$\tanh x$	$\ln(\cosh x)$
$-\mathrm{cosech}\ x \cdot \coth x$	$\mathrm{cosech}\ x$	$-\ln(\tanh \frac{1}{2}x)$
$-\mathrm{sech}\ x \cdot \tanh x$	$\mathrm{sech}\ x$	$2 \arctan(\exp x)$
$-\mathrm{cosech}^2 x$	$\coth x$	$\ln(\sinh x)$
$1/\sqrt{(x^2+1)}$	$\mathrm{arsinh}\ x$	$x\ \mathrm{arsinh}\ x - \sqrt{(1+x^2)}$
$1/\sqrt{(x^2-1)}$	$\mathrm{arcosh}\ x$	$x\ \mathrm{arcosh}\ x - \sqrt{(x^2-1)}$
$1/(1-x^2)$	$\mathrm{artanh}\ x$	$x\ \mathrm{artanh}\ x + \frac{1}{2}\ln(1-x^2)$
$-1/x\sqrt{(x^2+1)}$	$\mathrm{arcosech}\ x$	$x\ \mathrm{arcosech}\ x + \mathrm{arsinh}\ x$
$-1/x\sqrt{(1-x^2)}$	$\mathrm{arsech}\ x$	$x\ \mathrm{arsech}\ x + \arcsin x$
$1/(1-x^2)$	$\mathrm{arcoth}\ x$	$x\ \mathrm{arcoth}\ x + \frac{1}{2}\ln(x^2-1)$
$u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$	$u(x)\cdot v(x)$	$uv - \int v\dfrac{\mathrm{d}u}{\mathrm{d}v}\ \mathrm{d}v$
$\dfrac{1}{v}\dfrac{\mathrm{d}u}{\mathrm{d}x} - \dfrac{u}{v^2}\dfrac{\mathrm{d}v}{\mathrm{d}x}$	$\dfrac{u(x)}{v(x)}$	—
$r \exp(xa) \times \sin(\omega x + \phi + \theta)$	$\exp(ax) \times \sin(\omega x + \phi)$	$(1/r)\exp(ax)\sin(\omega x + \phi - \theta)$ $r = \sqrt{(\omega^2 + a^2)}\ \theta = \arctan(\omega/a)$

An ordered set of elements $\mathbf{x} = [x_1, x_2, x_3 \ldots x_n]$ is called an *n-vector*.

An $(n \times 1)$ matrix is called a *column vector* and a $(1 \times n)$ matrix a *row vector*.

1.2.10.2 Basic operations

If $\mathbf{A} = (a_{rs})$, $\mathbf{B} = (b_{rs})$,

(i) *Sum* $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is defined by $c_{rs} = a_{rs} + b_{rs}$, for $r = 1 \ldots m$; $s = 1 \ldots n$.

(ii) *Product* If $\mathbf{A}$ is an $(m \times q)$ matrix and $\mathbf{B}$ is a $(q \times n)$ matrix, then the product $\mathbf{C} = \mathbf{AB}$ is an $(m \times n)$ matrix defined by $(c_{rs}) = \sum_p a_{rp} b_{ps}$, $p = 1 \ldots q$; $r = 1 \ldots m$; $s = 1 \ldots n$. If $\mathbf{AB} = \mathbf{BA}$ then $\mathbf{A}$ and $\mathbf{B}$ are said to *commute*.

(iii) *Matrix-vector product* If $\mathbf{x} = [x_1 \ldots x_n]$, then $\mathbf{b} = \mathbf{Ax}$ is defined by $(b_r) = \sum_p a_{rp} x_p$, $p = 1 \ldots n$; $r = 1 \ldots m$.

(iv) *Multiplication of a matrix by a (scalar) element* If k is an element then $\mathbf{C} = k\mathbf{A} = \mathbf{A}k$ is defined by $(c_{rs}) = k(a_{rs})$.

(v) *Equality* If $\mathbf{A} = \mathbf{B}$, then $(a_{ij}) = (b_{ij})$, for $i = 1 \ldots n$; $j = 1 \ldots m$.

1.2.10.3 Rules of operation

(i) *Associativity* $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$, $\mathbf{A(BC)} = \mathbf{(AB)C} = \mathbf{ABC}$.

(ii) *Distributivity* $\mathbf{A(B+C)} = \mathbf{AB} + \mathbf{AC}$, $\mathbf{(B+C)A} = \mathbf{BA} + \mathbf{CA}$.

(iii) *Identity* If $\mathbf{U}$ is the $(n \times n)$ matrix (δij), $i, j = 1 \ldots n$, where $\delta ij = 1$ if $i = j$ and 0 otherwise, then $\mathbf{U}$ is the diagonal *unit matrix* and $\mathbf{A\,U} = \mathbf{A}$.

(iv) *Inverse* If the product $\mathbf{U} = \mathbf{AB}$ exists, then $\mathbf{B} = \mathbf{A}^{-1}$, the inverse matrix of $\mathbf{A}$. If both inverses $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist, then $(\mathbf{A\,B})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$.

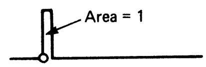
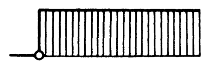
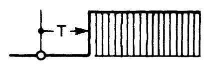
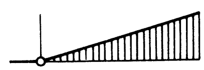
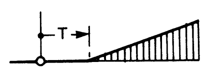
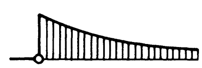
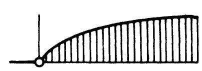
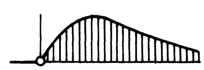
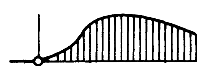
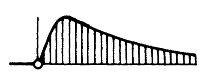
(v) *Transposition* The transpose of $\mathbf{A}$ is written as $\mathbf{A}^\mathbf{T}$ and is the matrix whose rows are the columns of $\mathbf{A}$. If the product $\mathbf{C} = \mathbf{AB}$ exists then $\mathbf{C}^\mathbf{T} = (\mathbf{AB})^\mathbf{T} = \mathbf{B}^\mathbf{T}\mathbf{A}^\mathbf{T}$.

(vi) *Conjugate* For $\mathbf{A} = (a_{rs})$, the congugate of $\mathbf{A}$ is denoted by $\mathbf{A}^* = (a_{rs}{}^*)$.

(vii) *Orthogonality* Matrix A is orthogonal if $\mathbf{AA}^\mathbf{T} = \mathbf{U}$.

Table 1.12 Laplace transforms

Definition	*f(t) from t = 0+*	$F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} f(t) \cdot \exp(-st) \cdot dt$
Sum	$af_1(t) + bf_2(t)$	$aF_1(s) + bF_2(s)$
First derivative	$(d/dt) f(t)$	$sF(s) - f(0-)$
*n*th derivative	$(d^n/dt^n) f(t)$	$s^n F(s) - s^{n-1} f(0-) - s^{n-2} f^{(1)}(0-) - \cdots - f^{(n-1)}(0-)$
Definite integral	$\int_{0-}^{T} f(t) \cdot dt$	$\frac{1}{s} F(s)$
Shift by T	$f(t-T)$	$\exp(-sT) \cdot F(s)$
Periodic function (period T)	$f(t)$	$\frac{1}{1-\exp(-sT)} \int_{0-}^{T} \exp(-sT) \cdot f(t) \cdot dt$
Initial value	$f(t),\ t \to 0+$	$sF(s),\ s \to \infty$
Final value	$f(t),\ t \to \infty$	$sF(s),\ s \to 0$

Description	*f(t)*	F(s)	*f(t) to base t*
1. Unit impulse	$\delta(t)$	1	
2. Unit step	$H(t)$	$\frac{1}{s}$	
3. Delayed step	$H(t-T)$	$\frac{\exp(-st)}{s}$	
4. Rectangular pulse (duration T)	$H(t) - H(t-T)$	$\frac{1-\exp(-sT)}{s}$	
5. Unit ramp	t	$\frac{1}{s^2}$	
6. Delayed ramp	$(t-T)H(t-T)$	$\frac{\exp(-sT)}{s^2}$	
7. *n*th-order ramp	t^n	$\frac{n!}{s^{n+1}}$	
8. Exponential decay	$\exp(-\alpha t)$	$\frac{1}{s+\alpha}$	
9. Exponential rise	$1 - \exp(-\alpha t)$	$\frac{\alpha}{s(s+\alpha)}$	
10. Exponential $\times t$	$t \exp(-\alpha t)$	$\frac{1}{(s+\alpha)^2}$	
11. Exponential $\times t^n$	$t^n \exp(-\alpha t)$	$\frac{n!}{(s+\alpha)^{n+1}}$	
12. Difference of exponentials	$\exp(-\alpha t) - \exp(-\beta t)$	$\frac{\beta - \alpha}{(s+\alpha)(s+\beta)}$	

cont'd

Table 1.12 (*continued*)

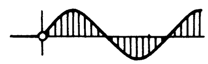
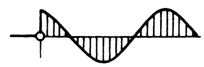
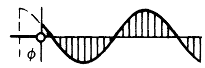
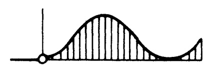
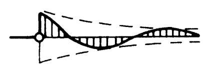
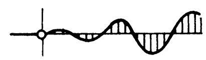
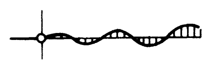
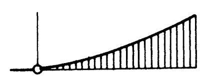
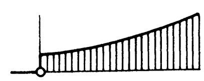
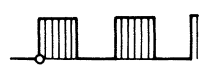
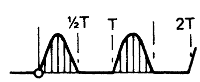
Definition	*f(t) from t = 0+*	$F(s) = L[f(t)] = \int_{0-}^{\infty} f(t) \cdot \exp(-st) \cdot dt$	
13. Sinusoidal	sin ωt	$\frac{\omega}{s^2 + \omega^2}$	
14. Phase-advanced sine	$\sin(\omega t + \phi)$	$\frac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$	
15. Sine × *t*	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$	
16. Exponentially decaying sine	$\exp(-\alpha t)\sin \omega t$	$\frac{\omega}{(s + \alpha)^2 + \omega^2}$	
17. Cosinusoidal	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	
18. Phase-advanced cosine	$\cos(\omega t + \phi)$	$\frac{s \cos \phi - \omega \sin \phi}{s^2 + \omega^2}$	
19. Offset cosine	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$	
20. Cosine × *t*	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$	
21. Exponentially decaying cosine	$\exp(-\alpha t)\cos \omega t$	$\frac{(s + \alpha)}{(s + \alpha)^2 + \omega^2}$	
22. Trigonometrical function G(*t*)	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$	
23. Exponentially decaying trigonometrical function	$\exp(-\alpha t) \cdot G(t)$	$\frac{2\omega^3}{[(s + \alpha)^2 + \omega^2]^2}$	
24. Hyperbolic sine	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$	
25. Hyperbolic cosine	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$	
26. Rectangular wave (period *T*)	*f*(*t*)	$\frac{1 + \tanh(sT/4)}{2s}$	
27. Half-wave rectified sine ($T = 2\pi/\omega$)	*f*(*t*)	$\frac{\omega \exp(sT/2)\operatorname{cosech}(sT/2)}{2(s^2 + \omega^2)}$	
28. Full-wave rectified sine ($T = 2\pi/\omega$)	*f*(*t*)	$\frac{\omega \coth(sT/2)}{s^2 + \omega^2}$	

Table 1.13 Decibel gain: power and voltage ratios

A	P_1/P_2	V_1/V_2	A	P_1/P_2	V_1/V_2
0	1.000	1.000	9	7.943	2.818
0.1	1.023	1.012	10	10.00	3.162
0.2	1.047	1.023	12	15.85	3.981
0.3	1.072	1.032	14	25.12	5.012
0.4	1.096	1.047	16	39.81	6.310
0.6	1.148	1.072	18	63.10	7.943
0.8	1.202	1.096	20	100.0	10.00
1.0	1.259	1.122	25	316.2	17.78
1.2	1.318	1.148	30	1000	31.62
1.5	1.413	1.189	35	3162	56.23
2.0	1.585	1.259	40	1.0×10^4	100.0
2.5	1.778	1.333	45	3.2×10^4	177.8
3.0	1.995	1.413	50	1.0×10^5	316.2
3.5	2.239	1.496	55	3.2×10^5	562.3
4.0	2.512	1.585	60	1.0×10^6	1 000
4.5	2.818	1.679	65	3.2×10^6	1 778
5.0	3.162	1.778	70	1.0×10^7	3 160
6.0	3.981	1.995	80	1.0×10^8	10 000
7.0	5.012	2.239	90	1.0×10^9	31 620
8.0	6.310	2.512	100	1.0×10^{10}	100 000

1.2.10.4 Determinant and trace

(i) The *determinant* of a square matrix **A** denoted by $|\mathbf{A}|$, also det(**A**), is defined by the recursive formula $|\mathbf{A}| = a_{11}\ \mathrm{M}_{11} - a_{12}\ \mathrm{M}_{12} + a_{13}\ \mathrm{M}_{13} - \cdots (\Leftarrow 1)^n a_{1n}\ \mathrm{M}_{1n}$ where M_{11} is the determinant of the matrix with row 1 and column 1 missing, M_{12} is the determinant of the matrix with row 1 and column 2 missing etc.
(ii) The *Trace* of **A** is denoted by $\mathrm{tr}(\mathbf{A}) = \Sigma_i\, a_{ii},\ i = 1, 2 \ldots n$.
(iii) *Singularity* The square matrix **A** is singular if det $(\mathbf{A}) = 0$.
(iv) The *Characteristic Polynomial* $\mathrm{P}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{U})$.

1.2.10.5 Eigensystems

(i) *Eigenvalues* The eigenvalues of a matrix $\lambda(\mathbf{A})$ are the n complex roots $\lambda_1(\mathbf{A}), \lambda_2(\mathbf{A}) \ldots \lambda_n(\mathbf{A})$ of the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{U}) = 0$. Normally in most engineering systems there are no equal roots so the eigenvalues are distinct.
(ii) *Eigenvectors* For any distinct eigenvalue $\lambda_i\,(\mathbf{A})$, there is an associated non-zero *right eigenvector* $\mathbf{X}_i$ satisfying the homogeneous equations $(\mathbf{A} - \lambda_i\mathbf{U})\ \mathbf{X}_i = \mathbf{0},\ i = 1, 2 \ldots n$. The matrix $(\mathbf{A} - \lambda_i\mathbf{U})$ is singular, however, because the det $(\mathbf{A} - \lambda_i\mathbf{U}) = 0$; hence $\mathbf{X}_i$ is not unique. In each set of equations $(\mathbf{A} - \lambda_i\mathbf{U})\ \mathbf{X}_i = \mathbf{0}$ one equation is redundant and only the relative values of the elements of $\mathbf{X}_i$ can be determined. Thus the eigenvectors can be scaled arbitrarily, one element being assigned a value and the other elements determined accordingly from the remaining non-homogeneous equations.

The equations can be written also as $\mathbf{AX}_i = \lambda_i\mathbf{X}_i$, or combining all eigenvalues and right eigenvectors, $\mathbf{AX} = \Lambda\mathbf{X}$, where Λ is a diagonal matrix of the eigenvalues and **X** is a square matrix containing all the right eigenvectors in corresponding order.

Since the eigenvalues of **A** and $\mathbf{A}^\mathbf{T}$ are identical, for every eigenvalue λ_i associated with an eigenvector $\mathbf{X}_i$ of **A** there is also an eigenvector $\mathbf{P}_i$ of $\mathbf{A}^\mathbf{T}$ such that $\mathbf{A}^\mathbf{T}\mathbf{P}_i = \lambda_i\mathbf{P}_i$. Alternatively the eigenvector $\mathbf{P}_i$ can be considered to be the *left eigenvector* of **A** by transposing the equation to give $\mathbf{P}_i^\mathbf{T}\mathbf{A} = \lambda_i\mathbf{P}_i^\mathbf{T}$, or combining into one matrix equation, $\mathbf{P}^\mathbf{T}\mathbf{A} = \mathbf{P}^\mathbf{T}\Lambda$.

Reciprocal eigenvectors Post-multiplying this last equation by the right eigenvector matrix **X** gives $\mathbf{P}^\mathbf{T}\mathbf{A}\,\mathbf{X} = \mathbf{P}^\mathbf{T}\Lambda\mathbf{X}$, which summarises the n sets of equations $\mathbf{P}_i^\mathbf{T}\mathbf{AX}_i = \mathbf{P}_i^\mathbf{T}\ \lambda_i\ \mathbf{X}_i = \lambda_i\mathbf{P}_i^\mathbf{T}\ \mathbf{X}_i = k_i\lambda_i$, where k_i is a scalar formed from the $(1 \times n)$ by $(n \times 1)$ vector product $\mathbf{P}_i^\mathbf{T}\ \mathbf{X}_i$. With both $\mathbf{P}_i$ and $\mathbf{X}_i$ being scaled arbitrarily, re-scaling the left eigenvectors such that $\mathbf{W}_i = (1/k_i)\ \mathbf{P}_i$, gives $\mathbf{W}_i^\mathbf{T}\ \mathbf{X}_j = \Leftarrow \delta ij = 1$, if $i = j$, and $= 0$ otherwise. In matrix form $\mathbf{W}^\mathbf{T}\mathbf{X} = \mathbf{U}$, the unit matrix. The re-scaled left eigenvectors $\mathbf{W}_i^\mathbf{T}$ are said to be the reciprocal eigenvectors corresponding to the right eigenvectors $\mathbf{X}_i$.
(iii) *Eigenvalue sensitivity analysis* The change in the numerical value of λ_i with a change in any matrix A element δa_{rs} is to a first approximation given by $\delta\lambda_i = (w_r)_i\ (x_s)_i\ \delta a_{rs}$ where $(w_r)_i$ is the r-th element of the reciprocal eignvector $\mathbf{W}_i$ corresponding to λ_i and $(\mathrm{x}_s)_i$ is the s-th element of the associated right eigenvector $\mathbf{X}_i$. In more compact form the sensitivity coefficients $\delta\lambda_i/\delta a_{rs}$ or condition numbers of all n eigenvalues with respect to all elements of matrix A are expressible by the 1-term dyads $\mathbf{S}_i = \mathbf{W}_i\ \mathbf{X}_i^\mathbf{T},\ i = 1 \ldots n$.

$$\mathbf{S}_i = \Leftarrow \begin{bmatrix} \delta\lambda_i/\delta a_{11} & \delta\lambda_i/\delta a_{12} & \ldots & \delta\lambda_i/\delta a_{1n} \\ \delta\lambda_i/\delta a_{21} & \delta\lambda_i/\delta a_{22} & \ldots & \delta\lambda_i/\delta a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & & \ldots & \\ \delta\lambda_i/\delta a_{n1} & \delta\lambda_i/\delta a_{n2} & \ldots & \delta\lambda_i/\delta a_{nn} \end{bmatrix}$$

The matrix $\mathbf{S}_i$ is known as i-th *eigenvalue sensitivity matrix*.
(iv) *Matrix functions* Transposed eigenvalue sensitivity matrices appear also in the dyadic expansion of a matrix and in matrix functions, thus $\mathbf{A} = \Sigma_i\lambda_i\mathbf{X}_i\ \mathbf{W}_i^\mathbf{T} = \Sigma_i\lambda_i\mathbf{S}_i^\mathbf{T}$, $i = 1 \ldots n$. Likewise $[\mathbf{A}]^2 > [\Sigma_i\lambda_i\ \ \mathbf{S}_i^\mathbf{T}]^2 = \Sigma_i\lambda_i^2\ \ \mathbf{S}_i^\mathbf{T}$ or in general $[\mathbf{A}]^\mathrm{p} = \Sigma_i\lambda_i^\mathrm{p}\ \ \mathbf{S}_i^\mathbf{T}$; thus, for example, $[\mathbf{A}]^{-1} = \Leftarrow \Sigma_i\ \lambda_i^{-1}\ \mathbf{S}_i^\mathbf{T}$.

1.2.10.6 Norms

(i) *Vector norms* A scalar measure of the magnitude of a vector $\boldsymbol{X}$ with elements $x_1, x_2 \ldots x_n$, is provided by a *norm*, the general family of norms being defined by $\|\boldsymbol{X}\| = [\Sigma_i\ |x_i|^p]^{1/p}$. The usual norms are found from the values of p.
If $p = 1$, $\|\boldsymbol{X}\|$ is the sum of the magnitudes of the elements,
$p = 2$, $\|\boldsymbol{X}\|$ is *Euclidean norm* or square root of the sum of the squares of the magnitudes of the elements,
$p =$ infinity, $\|\boldsymbol{X}\|$ is the *infinity norm* or magnitude of the largest element.
(ii) *Matrix norms* Several norms for matrices have also been defined, for matrix **A** two being the *Euclidean norm*, $\|\mathbf{A}\|_\mathrm{E} = [\Sigma_r\Sigma_s|\mathrm{a}_{rs}|^2]^{1/2},\ r = 1, 2 \ldots m;\ s = 1, 2 \ldots n$, and the *absolute norm*, $\|\mathbf{A}\| = \max_{r,s}\ |\ a_{rs}|$.

1.3 Physical quantities

Engineering processes involve energy associated with physical materials to convert, transport or radiate energy. As energy has several natural forms, and as materials differ profoundly in their physical characteristics, separate technologies have been devised around specific processes; and materials may have to be considered macroscopically in bulk, or in microstructure (molecular, atomic and subatomic) in accordance with the applications or processes concerned.

1.3.1 Energy

Like 'force' and 'time', energy is a unifying concept invented to systematise physical phenomena. It almost defies precise definition, but may be described, as an aid to an intuitive appreciation.

Energy is the capacity for 'action' or *work*.

Work is the measure of the change in energy *state*.

State is the measure of the energy condition of a *system*.

System is the ordered arrangement of related physical entities or processes, represented by a *model*.

Mode is a description or mathematical formulation of the system to determine its *behaviour*.

Behaviour describes (verbally or mathematically) the energy processes involved in changes of state. Energy *storage* occurs if the work done on a system is recoverable in its original form. Energy *conversion* takes place when related changes of state concern energy in a different form, the process sometimes being reversible. Energy *dissipation* is an irreversible conversion into heat. Energy *transmission* and *radiation* are forms of energy transport in which there is a finite propagation time.

In a physical system there is an identifiable energy input W_i and output W_o. The system itself may store energy W_s and dissipate energy W. The energy conservation principle states that

$$W_i = W_s + W + W_o$$

Comparable statements can be made for energy changes Δw and for energy rates (i.e. powers), giving

$$\Delta w_i = \Delta w_s + \Delta w + \Delta w_o \text{ and } p_i = p_s + p + p_o$$

1.3.1.1 Analogues

In some cases the *mathematical* formulation of a system model resembles that of a model in a completely different physical system: the two systems are then analogues. Consider linear and rotary displacements in a simple mechanical system with the conditions in an electric circuit, with the following nomenclature:

f	force [N]	M	torque [N m]	v	voltage [V]
m	mass [kg]	J	inertia [kg m^2]	L	inductance [H]
r	friction [N s/m]	r	friction [N m s/rad]	R	resistance [Ω]
k	compliance [m/N]	k	compliance [rad/N m]	C	capacitance [F]
l	displacement [m]	θ	displacement [rad]	q	charge [C]
u	velocity [m/s]	ω	angular velocity [rad/s]	i	current [A]

The force necessary to maintain a uniform linear velocity u against a viscous frictional resistance r is $f = ur$; the power is $p = fu = u^2 r$ and the energy expended over a distance l is $W = fut = u^2 rt$, since $l = ut$. These are, respectively, the analogues of $v = iR$, $p = vi = i^2 R$ and $W = vit = i^2 Rt$ for the corresponding electrical system. For a constant angular velocity in a rotary mechanical system, $M = \omega r$, $p = M\omega = \omega^2 r$ and $W = \omega^2 rt$, since $\theta = \omega t$.

If a mass is given an acceleration du/dt, the force required is $f = m(du/dt)$ and the stored kinetic energy at velocity u_1 is $W = \frac{1}{2}mu_1^2$. For rotary acceleration, $M = J(d\omega/dt)$ and $W = \frac{1}{2}J\omega_1^2$. Analogously the application of a voltage v to a pure inductor L produces an increase of current at the rate di/dt such that $v = L(di/dt)$ and the magnetic energy stored at current i_1 is $W = \frac{1}{2}Li^2$.

A mechanical element (such as a spring) of compliance k (which describes the displacement per unit force and is the inverse of the stiffness) has a displacement $l = kf$ when a force f is applied. At a final force f_1 the potential energy stored is $W = \frac{1}{2}kf_1^2$. For the rotary case, $\theta = kM$ and $W = \frac{1}{2}kM_1^2$. In the electric circuit with a pure capacitance C, to which a p.d. v is applied, the charge is $q = Cv$ and the electric energy stored at v_1 is $W = \frac{1}{2}Cv_1^2$.

Use is made of these correspondences in mechanical problems (e.g. of vibration) when the parameters can be considered to be 'lumped'. An ideal transformer, in which the primary m.m.f. in ampere-turns $i_1 N_1$ is equal to the secondary m.m.f. $i_2 N_2$ has as analogue the simple lever, in which a force f_1 at a point distant l_1 from the fulcrum corresponds to f_2 at l_2 such that $f_1 l_1 = f_2 l_2$.

A simple series circuit is described by the equation $v = L(di/dt) + Ri + q/C$ or, with i written as dq/dt,

$$v = L(d^2q/dt^2) + R(dq/dt) + (1/C)q$$

A corresponding mechanical system of mass, compliance and viscous friction (proportional to velocity) in which for a displacement l the inertial force is $m(du/dt)$, the compliance force is l/k and the friction force is ru, has a total force

$$f = m(d^2l/dt^2) + r(dl/dt) + (1/k)l$$

Thus the two systems are expressed in identical mathematical form.

1.3.1.2 Fields

Several physical problems are concerned with 'fields' having stream-line properties. The eddyless flow of a liquid, the current in a conducting medium, the flow of heat from a high- to a low-temperature region, are fields in which representative lines can be drawn to indicate at any point the direction of the flow there. Other lines, orthogonal to the flow lines, connect points in the field having equal potential. Along these equipotential lines there is no tendency for flow to take place.

Static electric fields between charged conductors (having equipotential surfaces) are of interest in problems of insulation stressing. Magnetic fields, which in air-gaps may be assumed to cross between high-permeability ferromagnetic surfaces that are substantially equipotentials, may be studied in the course of investigations into flux distribution in machines. All the fields mentioned above satisfy Laplacian equations of the form

$$(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) + (\partial^2 V/\partial z^2) = 0$$

The solution for a physical field of given geometry will apply to other Laplacian fields of similar geometry, e.g.

System	*Potential*	*Flux*	*Medium*
current flow	voltage V	current I	conductivity σ
heat flow	temperature θ	heat q	thermal conductivity λ
electric field	voltage V	electric flux Q	permittivity ε
magnetic field	m.m.f. F	magnetic flux ϕ	permeability μ

The ratio I/V for the first system would give the effective conductance G; correspondingly for the other systems, $q/\theta\varsigma$ gives the thermal conductance, Q/V gives the capacitance and Φ/F gives the permeance, so that if measurements are made in one system the results are applicable to all the others.

It is usual to treat problems as two-dimensional where possible. Several field-mapping techniques have been devised, generally electrical because of the greater convenience and precision of electrical measurements. For two-dimensional problems, conductive methods include high-resistivity paper sheers, square-mesh 'nets' of resistors and electrolytic tanks. The tank is especially adaptable to three-dimensional cases of axial symmetry.

In the electrolytic *tank* a weak electrolyte, such as ordinary tap-water, provides the conducting medium. A scale model of the electrode system is set into the liquid. A low-voltage supply at some frequency between 50 Hz and 1 kHz is connected to the electrodes so that current flows through the electrolyte between them. A probe, adjustable in the horizontal plane and with its tip dipping vertically into the electrolyte, enables the potential field to be plotted. Electrode models are constructed from some suitable insulant (wood, paraffin wax, Bakelite, etc.), the electrode outlines being defined by a highly conductive material such as brass or copper. The metal is silver-plated to improve conductivity and reduce polarisation. Three-dimensional cases with axial symmetry are simulated by tilting the tank and using the surface of the electrolyte as a radial plane of the system.

The conducting-*sheet* analogue substitutes a sheet of resistive material (usually 'teledeltos' paper with silver-painted electrodes) for the electrolyte. The method is not readily adaptable to three-dimensional plots, but is quick and inexpensive in time and material.

The *mesh* or resistor-net analogue replaces a conductive continuum by a square mesh of equal resistors, the potential measurements being made at the nodes. Where the boundaries are simple, and where the 'grain size' is sufficiently small, good results are obtained. As there are no polarisation troubles, direct voltage supply can be used. If the resistors are made adjustable, the net can be adapted to cases of inhomogeneity, as when plotting a magnetic field in which permeability is dependent on flux density. Three-dimensional plots are made by arranging plane meshes in layers; the nodes are now the junctions of six instead of four resistors.

A stretched elastic membrane, depressed or elevated in appropriate regions, will accommodate itself smoothly to the differences in level: the height of the membrane everywhere can be shown to be in conformity with a two-dimensional Laplace equation. Using a rubber sheet as a membrane, the path of electrons in an electric field between electrodes in a vacuum can be investigated by the analogous paths of rolling bearing-balls. Many other useful analogues have been devised, some for the rapid solution of mathematical processes.

Recently considerable development has been made in point-by-point computer solutions for the more complicated field patterns in three-dimensional space.

1.3.2 Structure of matter

Material substances, whether solid, liquid or gaseous, are conceived as composed of very large numbers of *molecules*. A molecule is the smallest portion of any substance which cannot be further subdivided without losing its characteristic material properties. In all states of matter molecules are in a state of rapid continuous motion. In a *solid* the molecules are relatively closely 'packed' and the molecules, although rapidly moving, maintain a fixed mean position. Attractive forces between molecules account for the tendency of the solid to retain its shape. In a *liquid* the molecules are less closely packed and there is a weaker cohesion between them, so that they can wander about with some freedom within the liquid, which consequently takes up the shape of the vessel in which it is contained. The molecules in a *gas* are still more mobile, and are relatively far apart. The cohesive force is very small, and the gas is enabled freely to contract and expand. The usual effect of heat is to increase the intensity and speed of molecular activity so that 'collisions' between molecules occur more often; the average spaces between the molecules increase, so that the substance attempts to expand, producing internal pressure if the expansion is resisted.

Molecules are capable of further subdivision, but the resulting particles, called *atoms*, no longer have the same properties as the molecules from which they came. An atom is the smallest portion of matter than can enter into chemical combination or be chemically separated, but it cannot generally maintain a separate existence except in the few special cases where a single atom forms a molecule. A molecule may consist of one, two or more (sometimes many more) atoms of various kinds. A substance whose molecules are composed entirely of atoms of the same kind is called an *element*. Where atoms of two or more kinds are present, the molecule is that of a chemical *compound*. At present over 100 elements are recognised (*Table 1.14*: the atomic mass number A is relative to 1/12 of the mass of an element of carbon-12).

If the element symbols are arranged in a table in ascending order of atomic number, and in columns ('groups') and rows ('periods') with due regard to associated similarities, *Table 1.15* is obtained. Metallic elements are found on the left, non-metals on the right. Some of the correspondences that emerge are:

Group 1a: Alkali metals
(Li 3, Na 11, K 19, Rb 37, Cs 55, Fr 87)
2a: Alkaline earths
(Be 4, Mg 12, Ca 20, Sr 38, Ba 56, Ra 88)
1b: Copper group (Cu 29, Ag 47, Au 79)
6b: Chromium group (Cr 24, Mo 42, W 74)
7a: Halogens (F 9, Cl 17, Br 35, I 53, At 85)
0: Rare gases
(He 2, Ne 10, Ar 18, Kr 36, Xe 54, Rn 86)
3a–6a: Semiconductors
(B 5, Si 16, Ge 32, As 33, Sb 51, Te 52)

In some cases a horizontal relation obtains as in the transition series (Sc 21 ... Ni 28) and the heavy-atom rare earth and actinide series. The explanation lies in the structure of the atom.

1.3.2.1 Atomic structure

The original Bohr model of the hydrogen atom was a central nucleus containing almost the whole mass of the atom, and a single *electron* orbiting around it. Electrons, as small particles of negative electric charge, were discovered at the end of the nineteenth century, bringing to light the complex structure of atoms. The hydrogen nucleus is a *proton*, a mass having a charge equal to that of an electron, but positive. Extended to all elements, each has a nucleus comprising mass particles, some (*protons*) with a positive charge, others (*neutrons*) with no charge. The atomic *mass number A* is the total number of protons and neutrons in the nucleus; the *atomic number Z* is the number of positive charges, and the normal number of orbital electrons. The nuclear structure is not known, and the forces that bind the protons against their mutual attraction are conjectural.