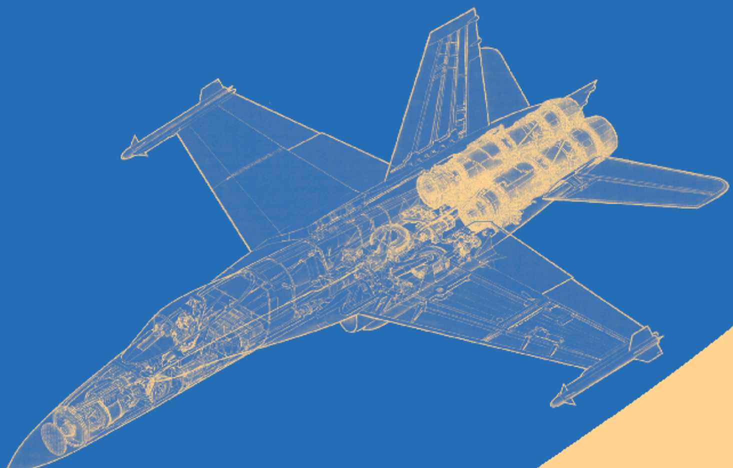
Analysis of Aircraft Structures

An Introduction

SECOND EDITION

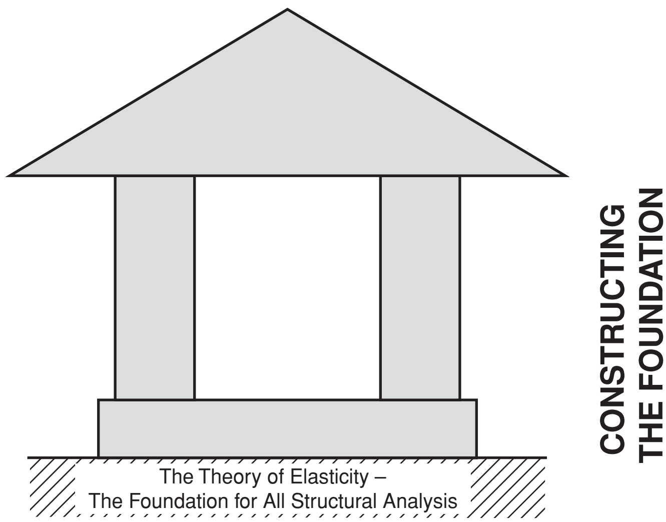
Bruce K. Donaldson



CAMBRIDGE AEROSPACE SERIES

PART I

THE FUNDAMENTALS OF STRUCTURAL ANALYSIS



I.1 An Overview of Part I

Vehicular weight, particularly that of aircraft and spacecraft, has a strong effect on the performance or economics of all such vehicles. Thus it is well worth spending many engineering man-hours on their design and analysis so as to make those vehicles as light-weight as possible. To make those many engineering hours of analysis as effective as possible, it is important that all the different types of loads that the vehicle will bear be well estimated, and then the structural response to those loads be carefully calculated. To carefully calculate the response of structures to estimated or measured loadings, it is important to use structural analysis techniques to which considerable confidence can be assigned. High degrees of confidence are achieved through experience and through thorough understanding of any approximations that are incorporated within the derivations of the selected structural analysis techniques. Thus it would seem that, in general, the fewer and the smaller the approximations, the more useful the structural analysis technique. This surmise is only partially true. As will be seen as the material of this textbook unfolds, the use of structural analysis techniques that contain essentially no approximations for many circumstances can be much too expensive and time consuming. Hence a compromise between cost and accuracy is necessary for good engineering practice. To understand how that compromise is found, this introduction to aerospace structures begins with the fundamentals of structural mechanics where the approximations are few in number and small in impact.

Part I of this textbook presents structural mechanics on a differential scale. That is, the focus of the analysis is typically on a volume of engineering material whose rectangular volume is $dx\,dy\,dz$. The enormous advantage of this approach is that the equations that are so established by this process apply to any type of component (beam, shell, solid) of any engineering structure simply because such a differential volume can be visualized as being part of the finite volume of that type of structural component. The frequent use of differential distances like dx should suggest that the calculus, a powerful analytic tool, figures prominently in Part I. The calculus is also vital to the remainder of the textbook because so much of that remainder is based upon the material of Part I. Not only is a knowledge of differential and integral calculus important, but certain other calculus-related aspects of mathematics should be well understood. The remaining four sections of this preface to Part I provide a review of those additional mathematical topics that are essential for a thorough understanding of the Part I material. Knowing the required mathematics makes the engineering much easier.

I.2 Summary of Taylor's Series

Let $f(x)$ be a function of single variable. The Taylor's series for $f(x)$ about $x = a$ may be written as

$$f(x) = f(a) + (x - a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \cdots$$

when all the derivatives exist and are continuous in a closed interval containing $x = a$. This same series is written in a slightly different style at the end of this section. The question of exactly when a Taylor's series is valid is not a simple one. The use of Taylor's series to represent the exceptionally smooth functions that generally describe stresses, strains, displacements, and the derivatives of these quantities in continuous structures has never led to contradictions. Hence this series is used freely whenever discontinuities are not suspected. A function that has a Taylor's series expansion is called "analytic."

In two dimensions, at $x = a$ and $y = b$, Taylor's series, written in a slightly different style, is

$$F(a + h, b + k) = F(a, b) + \sum \frac{1}{n!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^n F(x, y)\Big|_{x=a,y=b}$$

where the summation is from $n = 1$ to infinity. If the reader finds the style of presentation for the Taylor's series in two dimensions unfamiliar, it may help to note that, for example, the first series can be written in the style of the second series simply by substituting $(a + h)$ for the variable x. That is, where h is now the variable

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots$$

I.3 Summary of Newton's Method for Finding Roots

There are numerous approximate methods for finding the roots of polynomial equations, many of which are not limited to real roots. Newton's method is a simple matter when limited to real roots, and this method is not limited to polynomial equations. Newton's method is an iterative procedure, which means the same procedure is applied repeatedly until the results exhibit convergence to the degree of accuracy desired. Newton's method

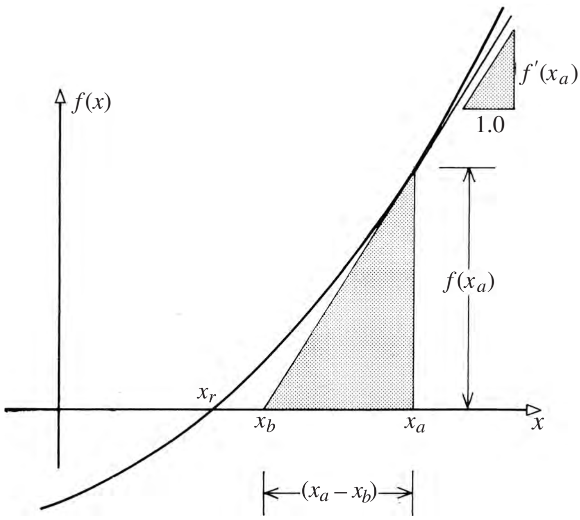


Figure I.1. A tangent (as opposed to secant) approach to determining the real roots of a single variable function.

begins with a first estimate for the location of the desired polynomial root, x_a. It does not matter how this initial estimate x_a is obtained. For example, the initial estimate of the root could be obtained from a rough graph of the polynomial equation. The first estimate is used in this iterative procedure to calculate a second estimate that is closer to the actual root, and the second estimate is used to calculate a still closer third estimate, and so forth. From Fig. I.1, it can be seen that from the interpretation of the derivative as a slope, $f'(x_a) = f(x_a)/(x_a - x_b)$. Solving this equation for the second estimate x_b yields

$$x_b = x_a - \frac{f(x_a)}{f'(x_a)}$$

Used repeatedly, this equation is the means of obtaining a series of improved estimates. The only caution is that the initial estimate has to be "close" enough to the desired root so that the process converges to that root. For example, if it were desired to discover the root $x = \pi$ of the equation $\sin x = 0$, then an initial guess of $x_a = 1$ would lead to the root $x = 0$ rather than the desired root.

See Refs. [44, 60] for a discussion of the intricacies of using Newton's method to find complex roots.

I.4 The Binomial Series

From Ref. [1], it may be proved via use of Taylor's series, that for any real number m, and for any x such that $|x| < 1$,

$$(1+x)^m = 1 + mx + m(m-1)\frac{x^2}{2!} + m(m-1)(m-2)\frac{x^3}{3!} + \cdots + [m(m-1)\cdots(m-n+1)]\frac{x^n}{n!} + \cdots$$

This series is only of finite length when m is equal to a positive integer.

I.5 The Chain Rule for Partial Differentiation

Consider a variable $q = Q(r, s, t)$. In this case, Q is an arbitrary function of the variables r, s, and t, which are called the first class variables. Let the first class variables be in turn functions of the second class variables x, y, and z; that is, $r = R(x, y, z)$, $s = S(x, y, z)$, and $t = T(x, y, z)$. Since the first class variables are dependent on the values of the second class variables, q can also be considered to be a function of the second class variables. Therefore derivatives of q can be taken with respect to the second class variables x, y, and z. The chain rule for partial differentiation of q with respect to x is as follows (Ref. [1]):

$$\frac{\partial q}{\partial x} = \frac{\partial Q}{\partial r}\frac{\partial R}{\partial x} + \frac{\partial Q}{\partial s}\frac{\partial S}{\partial x} + \frac{\partial Q}{\partial t}\frac{\partial T}{\partial x}$$

Notice the pattern of the variables. Each of the first class variables r, s, and t gets its chance to be part of a derivative of Q. Then each first class variable in turn is differentiated with respect to x, which is the second class variable with which the top variable q is differentiated in this illustration. The pattern for $\partial q/\partial x$ is that the leading function of each pair of products is always Q, the trailing variable is always x, and the connecting terms are always related to the first class variables.

The partial derivative of q with respect to y or z is the same as above but for x replaced by y or z. If the first class variables r, s, and t were only functions of x instead of x, y, and z, then $\partial q/\partial x$ would become dq/dx, and $\partial R/\partial x$ would become dR/dx, and so on. It is common practice to write $\partial r/\partial x$ in place of $\partial R/\partial x$, or dr/dx in place of dR/dx, and so on.

CHAPTER 1

Stress in Structures

1.1 The Concept of Stress

Structural engineers are concerned with the effects that forces produce on structures. That forces produce results such as deformations or structural collapse is the usual structural engineering cause-to-effect point of view. Even though this viewpoint is not the only possible or even useful viewpoint, it is the one adopted implicitly in Parts I, II, and III of this text as a temporary convenience until it becomes necessary to adopt a more general viewpoint. In other words, the usual engineering viewpoint is that the forces are an input, the structure is the system, and the effects of the forces acting on the structure (deformations, cracking, etc.) are the output. If a structural effect in turn influences the forces acting on the structure, then a feedback loop involving the forces and the structural effect exists. An example of structural feedback is first encountered in Part III of this text in the form of a beam buckling problem.

The theory that is developed in the next four chapters is valid for *any* type of force or combination of forces (within certain limits), and *any* type of structure. The task of classifying types of forces and structures can wait until it becomes necessary. What is necessary now is to begin to discuss the types of effects that forces produce on structures. One effect that forces can produce is *structural failure.* Structural failure is defined simply as occurring whenever a structure no longer can serve its intended use. A structural failure can be the dramatic collapse or rapid chain reaction disintegration of a large, enormously expensive structure (e.g., the Challenger space shuttle), or it can be as trivial as a wire clothes hanger being sufficiently bent out of shape that its usefulness as a clothes hanger is outweighed by the bother of straightening it. Clearly, certain structural failures are acceptable after an appropriate service life, and the service lives and performance of some structures have to be monitored or ended so as to avoid failures. The resulting question that structural engineers must address is the one that asks how structural failures can be anticipated with reasonable precision; that is, how can failures be predicted mathematically? In order to focus on the preliminary steps essential to predicting structural failures, this text omits discussion of the important topics of confirming mathematical predictions through testing or service experience monitoring.

The question of how to predict structural failures is a difficult question because there are many types of structural failure, and each type of failure has its own complexity. Returning to the example of wire clothes hangers, the large-deformation "bent-out-of-shape" type failure of the wire hanger to support three or more heavy winter coats is quite different from the fracture type of failure that results when a small portion of the hanger is repeatedly bent back and forth upon itself until the wire fractures. The process of predicting structural failures can be conveniently divided into two steps. The first step is the calculation of either or both of the analytical quantities called "stresses" and "displacements." (Definitions of stress and displacement are decided upon later.) The second step is to use, for example, the calculated stresses, the known material characteristics of the structure, and the loading characteristics to estimate the safety of the structure. This introductory text concentrates on

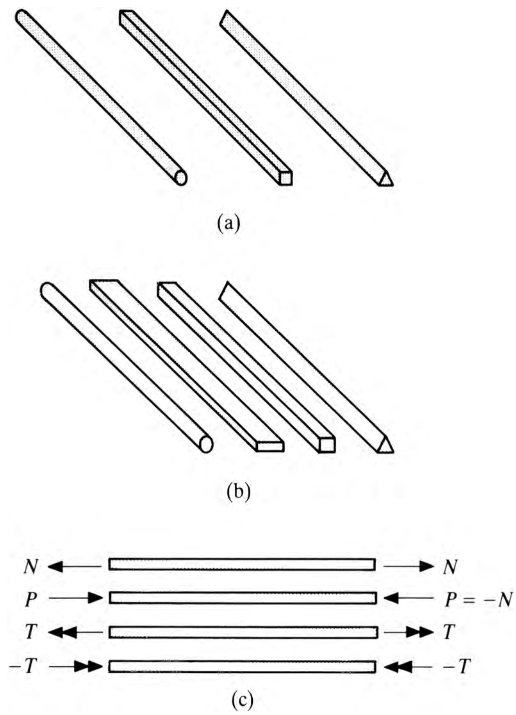


Figure 1.1. (a) Same length, uniform bars with the same cross-sectional area, but different cross-sectional shapes. (b) Same length, uniform bars with twice the cross-sectional area of the previous set. (c) By definition, "bars" only transmit axial forces (tensile or compressive) and twisting moments.

explaining the preliminary step of calculating stresses and displacements. Explanations of the process of using the calculated stresses or displacements to estimate the likelihood of failure is mostly left to more advanced texts, each of which generally concentrates on only one type of failure.

In this chapter the topic of stresses is introduced. The introduction is done in a complete manner that will not require extension or further elaboration short of the most advanced studies in solid mechanics. Thus this approach will save the reader time and effort in the process of learning the elements of structural mechanics. The first thing to be done is to provide an illustration of why engineers have developed the concept of stress, and the usefulness of that concept for determining when a structure will fall in a simple way. The same illustration will provide a basis for choosing a definition for stress. Consider the two sets of bars shown in Figs. 1.1(a) and 1.1(b). A bar is a long thin object of any constant cross-sectional shape that is subjected to only two types of loads. The first type of load is an axial force, that is, a force whose vector representation parallels an axis along the length of the bar. The second type of load is a twisting moment, also called a torque. Its double arrowhead vector representation (right-hand rule) is also one where the vector is parallel to an axis along the length of the bar. The conventional representations of bars loaded in the above manner are shown in Fig. 1.1(c). Let the bars in Fig. 1.1(a) all be well made from the same material and have the same cross-sectional area as that of a typical pencil. Let the bars in Fig. 1.1(b) have twice that cross-sectional area, and be well made of the same material as the bars in Fig. 1.1(a). If increasing tensile forces, that is, forces that tend to stretch the bars, are applied to each of the bars in Fig. 1.1(a), then it would be determined experimentally that all of the bars in Fig. 1.1(a) pulled apart (failed) at approximately the same final value of the applied-tensile force. The small differences between the magnitudes of the tensile forces at failure for each of the bars in Fig. 1.1(a) would be due to experimental measurement errors and small, unobserved differences between the bars. For cross-sectional areas like those of a typical pencil, or greater, it could also be discovered that the length of the bar has no appreciable effect on the magnitude of the failure load. Thus, from this first set of experimental results it can be concluded that, for this type of loading, the shape of the bar cross-section and its length are immaterial. Let the larger cross-section bars of Fig. 1.1(b) be subjected to the same experimental routine. Again it would be observed that each bar of this set failed at approximately the same final load value. Moreover, it would be observed that the failure loads for the bars of Fig. 1.1(b) are twice those of Fig. 1.1(a). In other words, doubling the cross-sectional area doubles the magnitude of the tensile failure load. Furthermore, this proportionality would continue for all larger and many smaller multiples of the cross-sectional area. (If the bar cross-sections are very small, for example, like those of thin wires, then small manufacturing imperfections may have large effects on the magnitude of the tensile load at failure.) Since a consistent goal of all engineers is to simplify their understanding of physical phenomena wherever possible, it is desirable to seek the best possible way to organize this simple data set. This can be done by noticing that the *one* thing that *all* the bars of Figs. 1.1(a) and 1.1(b) have in common is the ratio of the failure load value to the value of the cross-sectional area. The simple experiment described above suggests that the ratio of force to area is a primary means of predicting the behavior of structures and the materials from which they are made. Experiments with different materials, loadings and structural shapes would show that this conclusion is generally true. Hence a special name is bestowed on the ratio of force to area. The name is, of course, *stress*.

Very few useful structures are as simple as the bars in Fig. 1.1. No loadings are simpler than the tensile forces sketched in Fig. 1.1. The latter statement is based on the implication inherent in the sketch as it is drawn that the stress produced by the normal force, N, is evenly (i.e., "uniformly") distributed over the bar cross-sectional area, A. In other words, for this type of loading, the stress everywhere on the bar cross-section is equal to the average stress. In symbolic form, if σ_{av} is the average stress, then by the ratio concept of the preceding paragraph, $\sigma_{av} = N/A$. The question arises as to whether all stress distributions are necessarily uniform. To answer this question, consider two bars of equal length with equal-area square cross-sections. Let the first bar be made of rubber and let the second bar be made of steel. Let axial tensile forces be applied separately to each bar so as to stretch each bar exactly the same distance. The reader recognizes that a much greater force is required to stretch the steel bar the specified distance than is required to stretch the rubber bar that same distance. Thus, in these circumstances of equal areas, the average stress in the steel bar is much greater than the average stress in the rubber bar. Now consider the situation where the two bars in their equally stretched condition are bonded together to form one stretched bar. Clearly, from the viewpoint of the now single bar, the stress distribution is not uniform since the stress is much higher over the steel portion of the bar than over the rubber portion.

It is also true that a nonuniform stress distribution can exist over a cross-section of a bar made of only one material. Consider two square cross-section bars of cross-sectional area A which are made of the same material. Let the first of these two bars be loaded by a tensile force of magnitude 9000 lb, and the second by a tensile force of 1000 lb, where both forces are uniformly distributed over their respective cross-sections. Let the unloaded length of the first bar be just slightly and sufficiently shorter than that of the second bar so

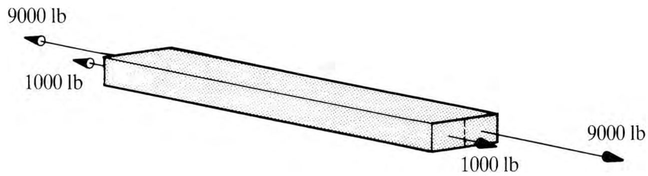


Figure 1.2. Illustration of the possibility of a nonuniform distribution of axial stresses across the cross-section of a bar.

that the stable loaded lengths of both bars are exactly the same. If the two bars are now fused together along their lengths to form one bar of cross-sectional area $2A$, while the respective loads are maintained on both halves of the now single bar, the result is a bar as shown in Fig. 1.2. In this case the bar is made of a single material with a stress acting over one half of the new cross-section that is nine times as great as that acting over the other half of the cross-section. When the stress does vary significantly, it is not useful to work with a value of the average stress over the entire cross-section. Clearly the more heavily loaded half of the fused bar is closer to rupture than the less loaded half. For that type of reason, engineers are usually much more interested in knowing the values of the peak stresses and knowing how extensive are the areas over which high stresses act. This latter information is much more useful when estimating the likelihood of local material failures or more general structural failures.

It is also important to note that simply stating that the two loads acting upon the combined bar's end cross-sectional area have a combined magnitude of 10 000 lbs would not be sufficiently informative with regard to the load distribution on the end surfaces of that bar. It will be necessary to be more precise when specifying the loads acting on the outer surfaces of the bodies under study.

The fact that stresses are not always constant over a given internal planar area requires careful consideration about how stress is to be defined. The definition must not compromise the basic concept of stress being the ratio of force over area. Since both force and area are measurable quantities, so then their ratio must also be a measurable quantity. Therefore it is necessary that the definition produce a unique measure; that is, that there be no ambiguity as to the magnitude of the stress. Consider Fig. 1.3(a). That sketch shows an edge view of a varying stress distribution acting over a beam cross-section that is positioned somewhere along the length of a loaded beam. (Beams have the same general geometry as bars, but the name "beam" indicates a more general type of loading.) Since an average stress value based on a total area is not a satisfactory measure, explore the possibility of a stress value that can be associated with smaller portions of the total area. A definition of stress based on sub-areas would permit having separate stress values where the stresses are high, or anywhere else. Therefore consider an arbitrarily located sub-area. Better yet, consider the arbitrarily located sequence of three sub-areas, from larger to smaller, as drawn in Fig. 1.3(b). Each sub-area in the sequence has been chosen as an included square, and each square has the point P as one vertex. Figure 1.3 (c) shows edge views of the stress distributions acting on that sequence of sub-areas. For the lack of a better idea, let it be said that the stress to be associated with each sub-area of the sequence is the average stress for that sub-area. Geometrically, the average stress is represented by the dashed line ordinate in Fig. 1.3(c). It is easily seen that in this case the value of the stress to be associated with each sub-area decreases as the area decreases. Similarly, if the sequence of squares approached point Q, instead of point P, the stress values would increase as the total force and the magnitude of the sub-area decreased. It also should be clear from the geometry that if the sequence of sub-area was greatly extended in an orderly fashion beyond three in number, the value of the average

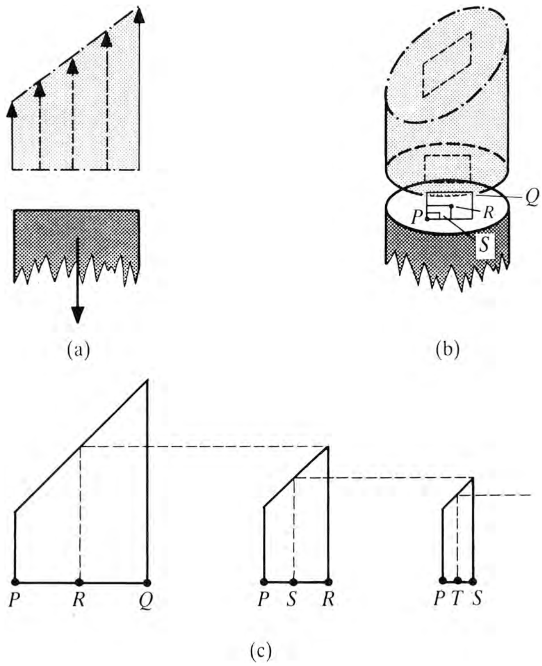


Figure 1.3. (a) Side view of a tensile force whose effect is distributed linearly over the bar cross-section. (b) The process of considering smaller and smaller portions of the bar cross-section at the fixed-point *P*. (c) A geometric illustration of how the average intensity of the distributed force near point *P* approaches a unique value as the small portion of the total cross-sectional area anchored at point *P* is systematically reduced by a factor of 4.

stress would stabilize (i.e., converge) as the sequence of dashed lines representing the stress magnitude approached either point *P* or *Q*. For example, for an approach to point *P*, the dashed line that depicts the average value of the stress over the sub-area in the sequence of sub-areas would irresistibly approach, and be confined by, the stress magnitude line at point *P*. This fixed stress magnitude at point *P* is precisely the unique force over area measure that is sought. This measure needs only to be expressed mathematically as the following limit,[1] where *N*, the total force acting over the sub-area *A*, is a function of *A*:

$$\text{Stress} \equiv \lim_{A \to 0} \frac{N}{A} \tag{1.1}$$

In this limit both the numerator and denominator decrease jointly to very small, even infinitesimal quantities.[2] Recall that the definition of a derivative is exactly the same type of limit. For example, the derivative of the function $f(x)$ at the point x_p is the limit as

[1] The three-bar symbol signifies that the relationship between the left-hand side and the right-hand side is that of an identity. An identity is an equality that is true in *all* circumstances. A simple way of appreciating the difference between an identity and a mere equality is to recall that for $0 \leq \theta \leq 2\pi$, the formula $\cos^2\theta + \sin^2\theta \equiv 1.0$ is true for all θ, while $\cos\theta + \sin\theta \equiv 1.0$ is only true for two values of θ. All definitions are identities.

[2] The atomic nature of materials is ignored in preference to the convenient fiction that all pure materials exhibit the same physical properties for small samples, no matter how small, as are exhibited on average for large samples of the material. This convenient approximation leads to the material being called a *continuum* and thus the material of Chapters 1–4 is called *solid mechanics*, a branch of continuum mechanics.

x approaches x_p of the ratio of $[f(x_p) - f(x)]/[x_p - x]$. Note how closely the above definition of the derivative fits the illustration in Fig. 1.3(c) where x_p is analogous to the point P, and x is analogous to the right-hand point in the sequence Q, R, S, ... This argument allows rewriting of the above definition, Eq. (1.1), as the ratio of two differentials; that is, as

$$\text{Stress} = \frac{dN}{dA} \tag{1.2}$$

This definition of stress is well and good as far as it goes, but it does not take into account the one further fact that the stress acting upon the cross-sectional area need not, as always assumed in the above discussion, act perpendicularly to the surface of the area under discussion. A simple demonstration that stress can act in the plane of the area as well as perpendicularly to the area occurs when one places one's hand firmly on a flat surface, and then rubs the surface with that hand, creating, by means of the friction between the hand and the surface, an in-plane stress on the flat surface. Note that the total force N acting upon the flat surface beneath the hand is the vector sum of the normal component (from pressing down with the hand) and the in-plane component. Since neither component is zero in this case, the total force vector N is neither normal to nor within the plane of the surface. Another confirmation of the possibility that the stress does not always act in a direction that is normal to the area under consideration can be obtained by merely passing an oblique plane through the first or second bar in Fig. 1.1(c). Since the total force, and hence the total stress, in the bar parallels the bar axis, and the normal to the oblique plane is not parallel to the bar axis, the stress is not normal to the oblique plane. Therefore, it is now necessary to adjust the above definition of stress to account for the directional properties of forces and stresses. When considering an area with a fixed orientation in space, stress is a vector quantity because it is a force vector (dN) divided by a scalar (dA). (The qualification "when considering an area with a fixed orientation" is important, and is developed later.)

On the basis of the above discussion, it is now a simple matter to define a *normal* stress as the limit of the ratio of the normal force acting upon an area of fixed orientation, as the magnitude of that area approaches zero. The same can be done for the in-plane stress, called the total *shearing stress.* This decomposition of the total stress into a normal stress and a total shear stress is significant because the effects of these two different types of stresses on materials can be quite different. Two more steps are necessary to make the above definitions still more useful. The first step is to introduce a coordinate system. To begin simply, consider a right-handed Cartesian[3] coordinate system where the x axis is normal to the area being studied, while the y and z axes lie in the tangent plane of the area under study. In this case the area is called an "x area," or "x surface," or "x plane," since the orientation of the plane is precisely located by its normal, the x axis. In other words, when the x axis is fixed in space, then any plane perpendicular to that axis is an x plane.

Lower-case sigma (σ) is chosen to symbolize stress. A double-subscript notation is used to identify which of the possible stress components is meant; see Fig. 1.4. The first subscript designates the plane of the area upon which the stress acts, while the second subscript designates the direction in which the stress acts. Looking at Fig. 1.4, for example,

$$\sigma_{xy} = \frac{dN_y}{dA_x} \tag{1.3}$$

[3] From Ref. [2], the adjective "Cartesian" is derived from the family name of René Descartes (1596–1650), who first introduced the coordinate method and established analytical geometry.

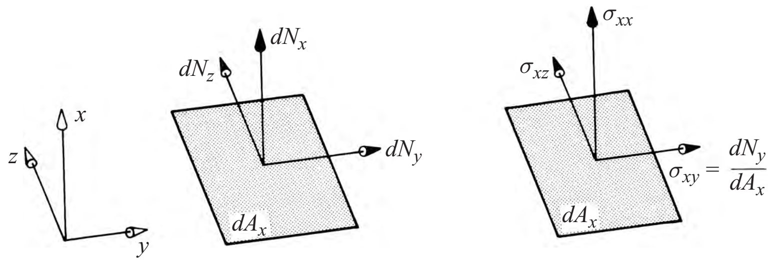


Figure 1.4. The total force acting upon a differential area, dA_x, can have no more than a differential magnitude, dN. Like any force vector, dN can be resolved into three Cartesian components. When these vector components are divided by the scalar value dA_x, the corresponding components of the total stress vector are obtained.

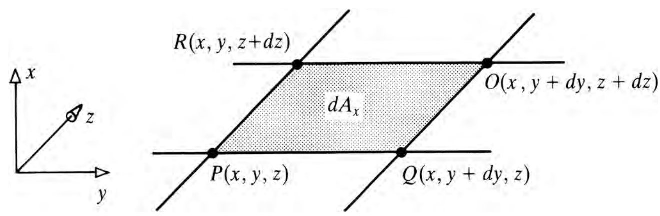


Figure 1.5. The bounds of the differential area dA_x that are always to be associated with the point P.

where dN_y is the y component of the total differential force acting over the differential area, dA_x. The double-subscript notation is illustrated more extensively below.

The second and final step to enhance the usefulness of the above definition of stress with its associated subscript protocols is to establish that stress, a quantity tied to an area, can be treated as an ordinary mathematical function of the coordinates of the chosen spatial coordinate sytem. A function that is solely a function of the spatial coordinates is called a "point function." Temperature is an example of a physical phenomena representable as a point function since a temperature value can be assigned to each point in the volume of the structure. In order to consider stress as a point function, let it be agreed that any point P in the structure can be identified by a set of Cartesian or other coordinates. In keeping with the double-subscript notation, let it also be agreed that the differential area that is associated with the stress definition at point P always lies in the plane defined by the coordinate axis of the first stress subscript. Let it be further agreed that the differential area always extends from point P in the direction of the two positive coordinate axes that parallel that plane, and the shape of the differential area is defined by constant values of the in-plane coordinates. Thus, for example, for Cartesian coordinates, an x plane differential area would have the shape of a rectangle of differential dimensions dy by dz as shown in Fig. 1.5. In this way, for all coordinate systems, a unique differential area consistent with the stress definition is automatically associated with any given point in the space of the structure. Therefore, by reversing the above viewpoint, the value of the stress acting on that differential area can be associated with a given point in the space of the structure. That is, it is now possible with clear meaning to speak, in terms of a coordinate system, about the stress or stresses "acting at a point." Therefore it is also possible to say that the stress components vary as the coordinates vary; or equivalently, that the stress is a function of the spatial coordinates. One result of the above convention is that it is now possible to understand clearly the meaning of a partial derivative of a stress component such as $\sigma_{xz}(x, y, z)$ with respect to a spatial coordinate such as y; that is, $\partial\sigma_{xz}/\partial y$.

Another way to suggest that such partial differentiation is valid is simply to note that (1) if, say, all the normal stresses acting on all the differential areas of a cross-section were added up (i.e., integrated), then the result would be the normal component of the total force acting across that cross-section; and (2) partial differentiation is the inverse operation of integration. Hence, if integration is valid, as it is since it is just the process of putting the stresses over the sub-areas together to form the total force N acting over the area, then the inverse operation of partial differentiation should also be possible for this physical phenomenon.

The next question to be addressed is: What is to be done with the various stress components? The present point of view is that the stresses arise as a result of external forces being applied to the structure under consideration. The stresses can be considered as the result of the forces being transmitted through, or equivalently, diffused throughout the structure. To illustrate this remark, think again of the first or second bar in Fig. 1.1(c). From simple static equilibrium considerations, if the bar is cut through anywhere in its interior, the same force that appears at an end will be found at the cut. In other words, the bar transmits the force from one of its ends to the other. The same *continuity of forces* exists in all loaded structures, and the force transmission requirement point of view is one starting point for the design of structural components. This concept of the continuity of forces, via internal stresses in this case, is but one aspect of Newton's laws of motion.[4] Newton's second law will now be used to quantify the above ideas.

1.2 The General Interior Equilibrium Equations

Newton's second law is applicable only to bodies of fixed mass. It is thus necessary to consider a structure of fixed mass that is completely enclosed by a fully specified outer boundary whose shape and size may change with time. To draw general conclusions using Newton's second law it is necessary to consider a structure of general shape. In this text, as in all others, bodies of general shape are always represented as having the shape of potatoes, potatoes with forces and moments acting upon them, with perhaps a hole or two. If there is a significant temperature change, (an important form of loading for vehicles of all kinds), then the body of general shape is a hot potato. Consider such a hot potato at a particular instant in time. Since the potato has no specific geometry, the only way available for discussing the stresses that exist in a potato in an unambiguous way is to examine the stresses acting upon infinitesimal areas. Hence consider an infinitesimal body in the interior of the general body. The infinitesimal body is chosen so that it is bounded by the distinct infinitesimal areas defined by the intersecting planes of the chosen coordinate system. To begin simply, a Cartesian coordinate system is first discussed. Hence, in this case, the infinitesimal areas are rectangles and the volume is a rectangular parallelepiped; see Fig. 1.6(a). The entire interior of the body up to, but not including, the boundary surface or surfaces (the skin of the potato) can be fashioned from such differential volumes.

The Cartesian components of the stresses that act upon this differential volume are shown from two different viewpoints in Figs. 1.6(b) and 1.6(c). The stress vector magnitudes are, as adopted previously, average values for the stress over their respective differential areas. Since the differential areas are so very small, the variation in the stress magnitude is very

[4] According to Ref. [2], Sir Isaac Newton (1642–1727) published the laws of motion in his famous *Philosophiae Naturalis Principia Mathematica* in 1687.

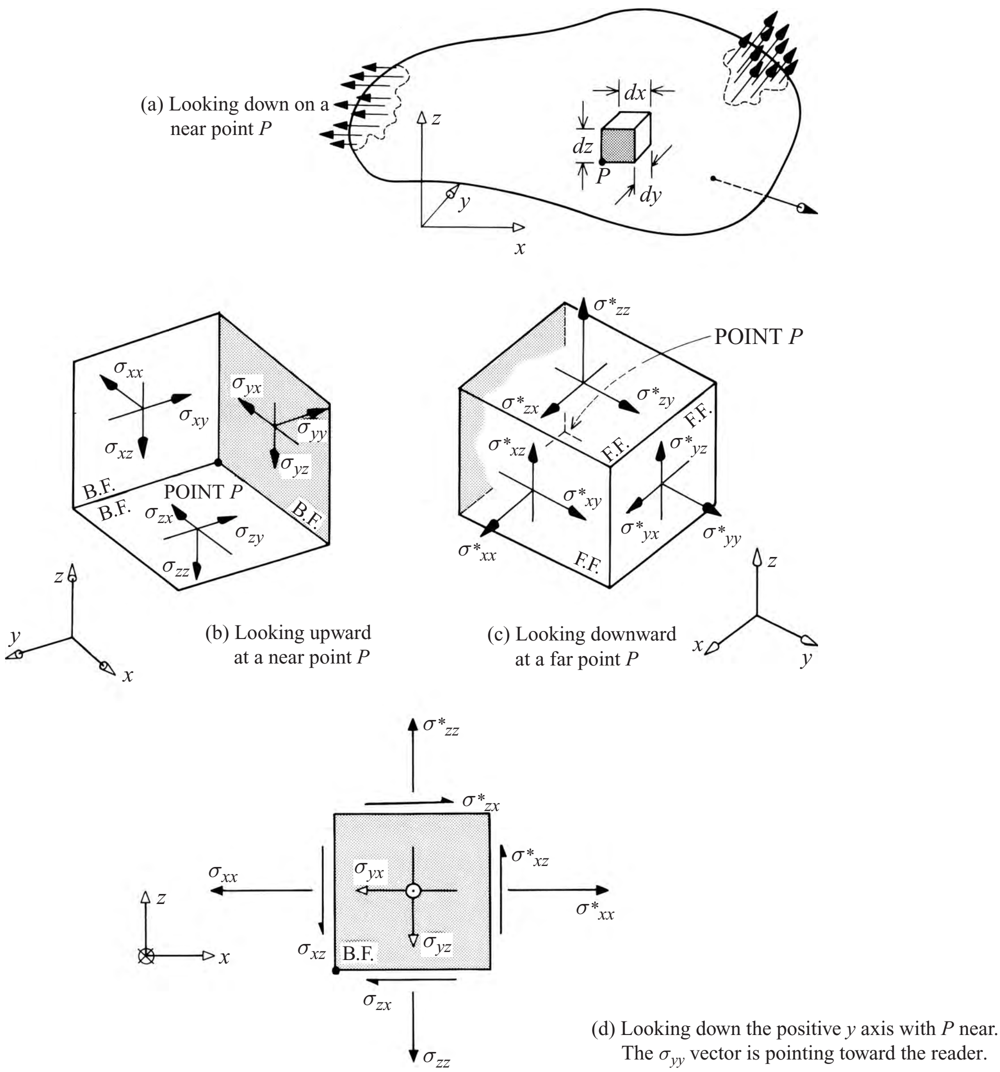


Figure 1.6. (a) A structural body of general shape subjected to any loadings acting upon its surface and any body forces acting throughout its volume. In accord with the selected coordinate system, there is a unique differential volume extending in the positive coordinate axis directions from the arbitrarily selected point P. When the coordinate system is Cartesian, the differential volume has the shape of a rectangular parallelepiped. (b) A view of the rectangular parallelepiped faces, called "back faces," temporarily abbreviated as B.F., that intersect at point P. To help interpet the orientation of these rectangular parallelepipeds, in all four different views of this same parallelepiped, the back y face is shaded. (c) A view of the three "front faces" (F.F.) associated with the point P. In order to distinguish between the corresponding stresses on the front and back faces, an asterisk is *temporarily* added to the front face stress symbols. (d) A side view of the same parallelepiped showing the back y face. Note the convention that the arrowheads of vectors coming directly out of the paper appear as small circles with a dot at the center, while vectors into the paper are symbolized by small crosses (the feathers) centered on small circles.

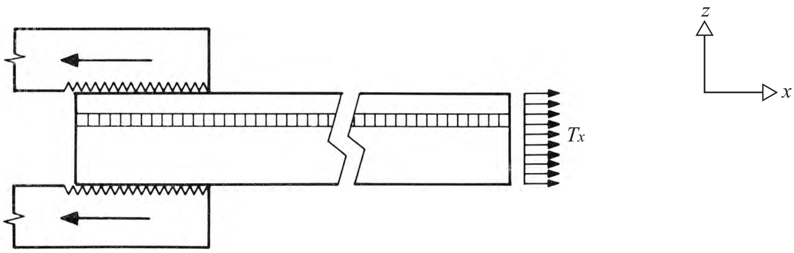


Figure 1.7. A test specimen gripped on one end and loaded upon the other end by a uniform traction.

close to being planar (a plane at a tilt). Therefore the average value of the stress magnitude occurs very close to the centroid of the differential area: in this case, at the center of each rectangular face. Figure 1.6(d), a side view of the same infinitesimal element, ties together the three faces shown in Fig. 1.6(b) with the other three faces shown in Fig. 1.6(c). Note that the point *P* in Fig 1.6(b) is the corner of the rectangular parallelepiped that has the lowest coordinate values. Accordingly, by the convention adopted above for associating the areas over which the stresses act with a nearby point, the stresses acting upon the faces shown in Fig. 1.6(b) are the stresses "acting at point *P*." The three faces of the rectangular parallelepiped that intersect at point *P* are called "back faces." On the back faces the standard convention for positive stress vectors is that they are always oppositely directed from the positive coordinate axes that they parallel. The front faces shown in Fig. 1.6(c) have higher coordinate values than the corresponding back faces, and the front face stress vectors are always positive in the positive coordinate axis direction. Note again the pattern of the double subscripts where, for both front and back faces, the first subscript identifies the (x, y, or z) face upon which the stress acts while the second subscript identifies the axis paralleled by the stress vector. The reader must memorize the stress sign convention and be able to apply the double subscript notation to differential elements whose geometry is defined by all common orthogonal coordinate systems, and do so from any viewing angle.

The stresses on the front faces are distinguished from those acting on the back faces by the temporary addition of an asterisk to signify that the front face stresses may be different from the stresses with the same subscripts that act on the back faces. To show that the two sets of stresses can be different, examine Fig. 1.7. At this point the term "tractions" will be introduced to refer to stresses acting on the outside of any outer boundary surface. In other words, stresses and tractions are exactly the same thing, but stresses occur inside the structural body, or on the inside of outer surface boundaries, while tractions occur on the outside of the outer surface boundaries.[5] (A sign convention for tractions is introduced later.) The bar in Fig. 1.7 is being stretched by a uniformly distributed traction, T_x, acting on the right-hand end face of the bar. The left end of the bar is being gripped by the jaws of a clamp. The clamp equilibrates the resulting force applied to the right-hand end by producing oppositely directed shearing tractions on the cylindrical outer surface of the bar. Now consider the long row of infinitesimal rectangular parallelepipeds that runs parallel to the bar axis from the bar's left-hand end face to the bar's right-hand end face as is shown in Fig. 1.7. Ignoring the insignificant effect of air pressure in this situation, there is no traction acting on the left-hand face of the furthermost left-hand infinitesimal element in the row of such elements. Isolating and internalizing this infinitesimal element, (so that the tractions can be referred to as stresses), it then can be said that there is zero stress on that left-hand

[5] The term "surface boundaries" is meant to include the boundaries of any cavities within the general body.

face of the left-hand element. Applying the same sort of reasoning to the isolated right-hand element, it can be said that the stress acting on the right-hand face of the right-hand element has the magnitude of the applied traction, T_x. Thus over the finite length of the bar, there is a finite change in the stress σ_{xx} from a value of zero at one end to a value of T_x at the other end. Thus it is necessary to conclude that, for at least some of these infinitesimal rectangular parallelepipeds, there is a change of some sort in the values of the σ_{xx} stress from that at their left-hand face to that at their right-hand face. Thus, by extending the same sort of argument to each of the other stresses, it may be concluded that the temporary asterisks applied to the stresses on the front faces of the generally stressed rectangular parallelepiped are appropriate for making a distinction between the corresponding back face and front face stresses.

Figures 1.6(b) and 1.6(c) show a count of 18 different stresses (nine with asterisks and nine without) before application of Newton's second law. It is quite possible to proceed using those 18 quantities when summing forces in, and taking moments about, the three orthogonal directions. Summing forces and moments in that manner produces six equations relating the 18 quantities. There is a much better way of representing the stresses with asterisks so as to produce more manageable, albeit differential, equations of equilibrium. One important advantage of this second approach is that the difference between the number of unknown stresses and the number of equations that relate these stresses is only 3 as opposed to 12 when using asterisks. To begin this second approach, consider the stress σ_{xx} on the back face, and the stress σ^*_{xx} on the front face, of the rectangular parallelepiped: again see Fig. 1.6(d). Consider also the stress in the x direction on the back face of a neighboring rectangular parallelepiped of the same size lying to the right of the first parallelepiped. The back face of the neighboring parallelepiped abuts the front face of the first parallelepiped. Newton's third law concerning equal and opposite reactions establishes that the stress in the x direction on the neighboring back face is σ^*_{xx}. Then a comparison of these back face stresses in the x direction of these two adjacent elements shows that the stress values σ_{xx} and σ^*_{xx} are values of the same stress function at different points in space separated by a differential distance. Therefore these two values of the same stress function can be related by means of a Taylor's series expansion[6] when it is assumed that the stress function varies in such a smooth fashion that it has an infinite number of derivatives. Again, a short summary of Taylor's series is presented in Section I.2. Let $\sigma_{xx}(x, y, z)$ symbolize the stress function. When expanding this function about the x value associated with the back face at P (which is simply called x) to obtain the value of σ_{xx} at the front face, a distance dx from the back face, only the variable x undergoes a change from the value x to the value $x + dx$. Thus the Taylor's series has the form

$$\sigma_{xx}(x + dx, y, z) = \sigma_{xx}(x, y, z) + dx\left(\frac{\partial \sigma_{xx}}{\partial x}\right) + \left(\frac{1}{2!}\right)(dx)^2\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2}\right) + \left(\frac{1}{3!}\right)(dx)^3\left(\frac{\partial^3 \sigma_{xx}}{\partial x^3}\right) + \cdots \tag{1.4}$$

[6] According to Ref. [1], Taylor's series was first published in 1715 by Brook Taylor. Taylor's formula with remainder was derived by Lagrange in 1797, and later by Ampére in 1806. The assumption of the validity of the Taylor series is not really necessary in this case. The same final result can be obtained from the simple geometric interpretation of the first derivative as a tangent that well approximates the function over infinitesimal distances. The Taylor's series argument was chosen for presentation in place of the simpler geometric argument because only the former argument makes clear the error associated with retaining only the first and second terms in the expansion. The expansion shows that the error is infinitesimal.

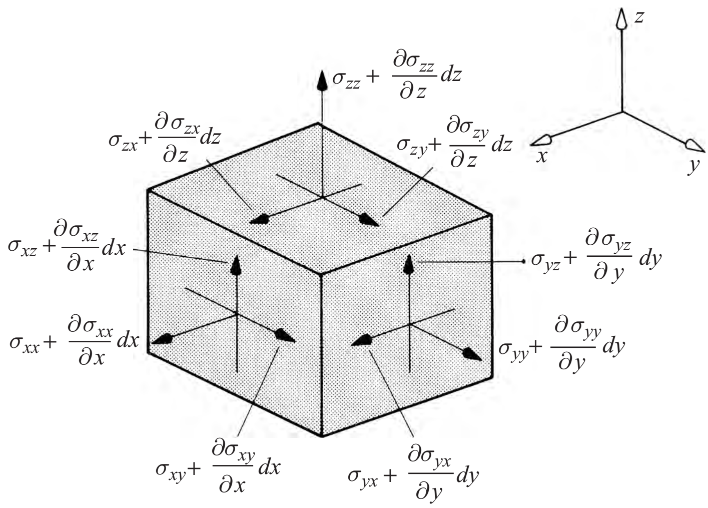


Figure 1.8. The three front faces of a rectangular parallelepiped and the stress vector components that always act upon those faces. The presence of the infinitesimal changes in the stress values, that is, changes from those values on the back faces, renders obsolete the previous use of asterisks to denote front face stresses.

where, by the previous notation

$$\sigma^*_{xx} = \sigma_{xx}\,(x + dx,\, y,\, z)$$

Once again, note from Fig 1.6(d) that the values of the coordinate y and z are unchanged when moving from the center of the back face to the center of the front face. Hence partial derivatives with respect to y and z, which represent changes in these directions, are zero, and do not appear in the above expansion. Examination of Eq. (1.4) shows that the difference between σ^*_{xx}, on the front face and σ_{xx} on the back face is a series of terms each of which includes an infinitesimal differential factor dx, with each succeeding term including an additional infinitesimal factor dx. Thus each succeeding term is infinitesimal compared to the preceding term. Thus all the terms after the second term are certainly negligible compared to the second term, and these terms are dropped from further consideration. Even though it is also true that the second term is infinitesimal compared to the first, it is not dropped because if it were dropped, then the entire difference over the infinitesimal distance dx between σ^*_{xx} and σ_{xx} would be lost, and that cannot be permitted without losing the possibility of coping with finite differences over finite distances. In other words, the second term is by far the largest part of the difference between the stresses at the two faces.

The final result, following the same development for the other stresses with asterisks as was done for σ^*_{xx} above, is that all the stresses with the temporary asterisks are now replaced by the two-term sums shown in Fig. 1.8. Now it is possible to apply Newton's second law and obtain manageable equations. The style of applying Newton's second law used here may require a brief explanation. Note that the usual form, where boldface terms are vector quantities, is

$$\sum \mathbf{F}_i = m\mathbf{a} \quad (i = 1, n)$$

where of course, the $\mathbf{F}_i$ are the various forces acting upon the body of mass m, and $\mathbf{a}$ is the total acceleration of the body. This identity may be rewritten as

$$\sum \mathbf{F}_i - m\mathbf{a} = 0 \quad (i = 1, n)$$

By choosing to define the term $-m\mathbf{a}$ as a force called the "inertia force," which is possible since it too is a vector with units of force, the inertia force can be included in the summation with all the other forces. Then, in these circumstances, Newton's second law has the appearance

$$\sum \mathbf{F}_i = 0 \quad (i = 1, n + 1) \tag{1.5}$$

This latter equation is the equation of "dynamic equilibrium" when inertia forces are actually included in the sum, that is, when either the magnitudes or directions of the accelerations are functions of time. It is the familiar equation of "static equilibrium" when all included accelerations are constants. This latter, yet all-inclusive, form of Newton's second law is used whenever it is desired to sum forces and moments and not make a distinction between static and dynamic equilibrium. In other words, adopting the inertia force point of view is simply a means of retaining the generality necessary to meet the problems of structural dynamics, yet at the same time not divert attention from the focus of this text, which is on structural mechanics.

Inertia forces (the negative of mass multiplied by acceleration) are tied to the mass of the body. The reader has probably experienced on his or her body the thrilling effects of the inertia forces that result from the accelerations imposed by the convoluted paths followed by carnival rides. A simpler example for inertia forces is provided by a high-speed elevator in a tall building. When the elevator accelerates upward, a passenger feels his or her entire body being pulled downward. When the elevator decelerates (i.e., has a negative acceleration), the passenger feels almost able to float. Such forces that act over the mass of a body are simply called "body forces." It is irrelevant whether or not the accelerations vary with time as in the above-cited examples, or are constant as in the case of weight forces. The vector sum of all the body forces is the total body force, and the magnitudes of its three Cartesian components per *unit mass* (which are just the total acceleration components) are identified herein as B_x, B_y, and B_z. Similar designations are used with other coordinate systems. All such components are positive in the direction of the positive coordinate axes.

Now, at long last, everything is in place to write the equations of equilibrium for the infinitesimal element of Fig. 1.8 whose geometry is defined by the choice of a Cartesian coordinate system. Note that the sketch of the infinitesimal element constitutes a free body diagram, and this is the only free body diagram that can be drawn for a body of general shape. The reader is cautioned that it is absolutely necessary to draw free body diagrams before summing forces and moments. Begin by summing forces in the x direction. Since Newton's second law involves forces, not stresses, each stress has to be multiplied by the area over which it acts to obtain the corresponding force. Similarly, the body force components per unit mass need to be multiplied by the body mass, that is, mass density symbolized by lowercase rho, ρ, multiplied by body volume. Then

$$\begin{aligned}
&-\sigma_{xx}\,dy\,dz + \left[\sigma_{xx} + \left(\frac{\partial \sigma_{xx}}{\partial x}\right) dx\right] dy\,dz \\
&-\sigma_{yx}\,dx\,dz + \left[\sigma_{yx} + \left(\frac{\partial \sigma_{yx}}{\partial y}\right) dy\right] dx\,dz \\
&-\sigma_{zx}\,dx\,dy + \left[\sigma_{zx} + \left(\frac{\partial \sigma_{zx}}{\partial z}\right) dz\right] dx\,dy + \rho B_x\,dx\,dy\,dz = 0
\end{aligned}$$

The above identity can be simplified by canceling terms of opposite sign, and dividing through by the quantity $dx\ dy\ dz$. Such a division is permissible because, although each of these differentials is made to approach zero, they are not permitted to be zero. The result is

$$\boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho B_x = 0} \tag{1.6a}$$

The reader can readily verify that the summation of forces in the y and z directions leads to the following two additional identities

$$\boxed{\begin{aligned} \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho B_y = 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho B_z = 0 \end{aligned}} \tag{1.6b}$$

Now sum moments about an axis parallel to the z axis that passes through the center of the parallelepiped. The advantage of placing the axis at the center of the differential element is that all of the body forces and all the forces associated with the normal stresses have zero moment arms. In addition, the shear stresses on the z faces also have zero moment arms. The shear stresses on the x and y faces that have z subscripts parallel the z axis and thus also do not result in a moment about the z axis. Furthermore, assume there are no body moments such as might result from a magnetic field acting upon a magnetic material.[7] The sum of the moments therefore reduces to

$$\begin{aligned} &+\sigma_{xy}\, dy\, dz \left(\frac{1}{2}dx\right) + \left[\sigma_{xy} + \left(\frac{\partial \sigma_{xy}}{\partial x}\right) dx\right] dy\, dz \left(\frac{1}{2}dx\right) \\ &-\sigma_{yx}\, dx\, dz \left(\frac{1}{2}dy\right) - \left[\sigma_{yx} + \left(\frac{\partial \sigma_{yx}}{\partial y}\right) dy\right] dx\, dz \left(\frac{1}{2}dy\right) = 0 \end{aligned}$$

Dividing through by the quantity $dx\ dy\ dz$ yields

$$+\sigma_{xy} - \sigma_{yx} + \frac{1}{2}\left(\frac{\partial \sigma_{xy}}{\partial x}\right) dx - \frac{1}{2}\left(\frac{\partial \sigma_{yx}}{\partial y}\right) dy = 0$$

The latter two terms each contain an infinitesimal differential factor. Thus they are infinitesimal compared to the first two terms and are accordingly dropped.[8] If inertial torques, that is, torques due to angular accelerations, were included in this sum of moments acting upon the differential rectangular parallelepiped, the inertial torques would initially involve five

[7] The astute reader recalls that the stress profile over any infinitesimal area is, in general, planar. Since it is not a constant, there is a moment on each surface area due to the variation in the stress distribution. However, this type of moment involves four differential factors. Thus it, like the other partial derivative terms in the moment summation, makes only a negligible contribution to the total summation. See the exercises at the end of this chapter for an illustration of this point.

[8] Note that for the sole purpose of deriving these moment equations, the differential element in Fig. 1.8 could have been drawn without the stress magnitudes on the front faces including the additional differential terms that distinguish the front face stresses from the back face stresses. The reader is advised not to be confused by the fact that it is not unusual for basic engineering textbooks dealing with the mechanics of fluid flow, heat transfer, and structural mechanics to omit such differential increments *whenever* these differential increments, as in this case of the moment equations, have no effect on the equation being derived. It should be remembered that in general, whether or not they contribute to the final equation, differential increments are always actually present on differential elements.

differential factors. Hence they too would be negligible. The final result for this summation, and similarly for the other two summations, is

$$\boxed{\sigma_{xy} = \sigma_{yx} \quad \sigma_{xz} = \sigma_{zx} \quad \sigma_{yz} = \sigma_{zy}} \tag{1.7}$$

The happy conclusions from the moment equilibrium equations are that there are only six ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz},$ and σ_{yz}) rather than nine distinct stresses with which it is necessary to contend, and the order of the stress subscripts is immaterial. These two conclusions are incorporated in all the work that follows.Therefore there is no need to refer to Eqs. (1.7) again, and Eqs. (1.6) alone are henceforth referred to as the *equilibrium equations*. Since Newton's second law applies to any engineering structure, Eqs. (1.6) always apply everywhere in the interior of a structure of any shape made of any material subjected to any mechanical and thermal loading.

1.3 Equilibrium at the Outer or Inner Boundary

It is now necessary to examine the boundary of the structural body of general shape on the same infinitesimal scale as was used for the interior of this structural body. The purpose is to discover what conditions on the stresses result from requiring that a differential mass at the outer boundary of the general body be in a state of equilibrium. In parallel to the development for the interior of the general body, the first thing to be done here is to deal with the geometry of a general body boundary. It is easy to see that any closed curve that lies in a plane can be approximated with any degree of desired accuracy by joining smaller and smaller secants in the same fashion that a given circle can be approximated by an inscribed regular polygon as the number of polygonal sides increases without bound. In three dimensions, in a similar fashion, a surface in three-dimensional space can be approximated to any degree of desired accuracy by connecting planar triangles as the triangles are made smaller and smaller. Hence, to examine a portion of a general surface is to examine an infinitesimal plane triangle that is as close an approximation to that surface as desired. Again, since Newton's second law is for bodies rather than surfaces, it is necessary to form a body which has the infinitesimal triangle of the boundary surface as one of its distinct surfaces. When Cartesian coordinates are used, the body to be formed requires x, y, and z faces as well as the triangular area of the boundary surface. The body that meets these requirements is simply an irregular tetrahedron as shown in Fig. 1.9(a).

It is now necessary to introduce a sign convention for the tractions acting on the boundary. Let the tractions be positive in the positive coordinate directions regardless of the boundary surface on which they act. Recall that just as the total stress vector acting on a given internal surface can be at any angle to that surface, the total traction vector can have any angular orientation with respect to the boundary surface. Thus the components of the total traction vector are wholly independent of each other. Figure 1.9(b) illustrates the sign convention for the tractions in Cartesian coordinates. The tetrahedron's interior surfaces bear the stresses shown because they are back faces; that is, they pass through the point with the lowest coordinate values, point P.

As an aside, note that on the infinitesimal scale being discussed here, even a "concentrated force" acting upon an outer surface is just an intense traction. Indeed, a concentrated force is just a convenient mathematical fiction for an intense traction acting over a very small area compared to the total area of the body under discussion. That is, no force, finite or infinitesimal, can act upon a single point because a point has a zero area. The pressure (i.e., traction) would be infinite, and thus beyond engineering experience.