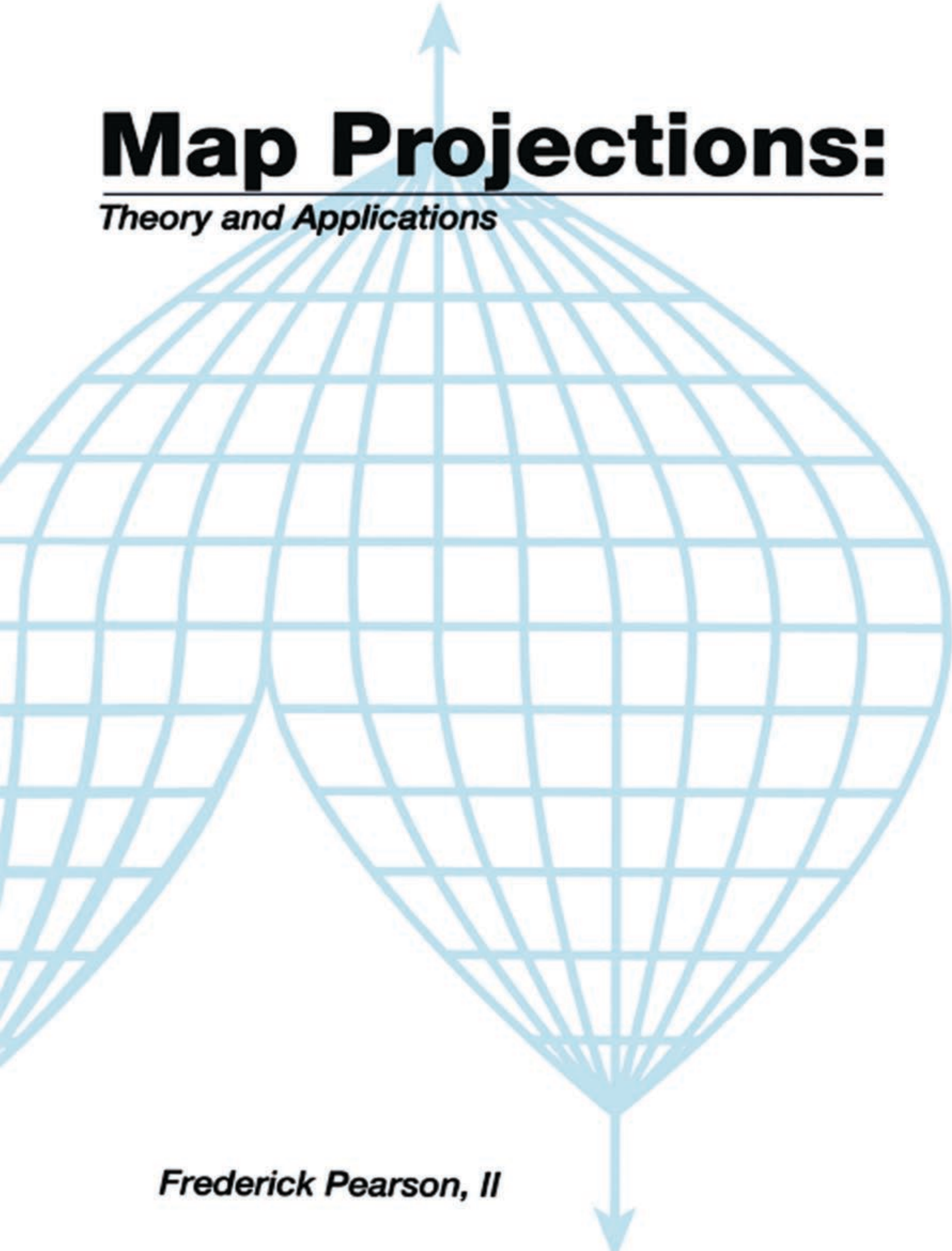
Map Projections:

Theory and Applications



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Chapter 2

MATHEMATICAL FUNDAMENTALS

I. INTRODUCTION

The process of map projection requires the transformation from a set of two independent coordinates on the model of the Earth to a set of two independent coordinates on the map. The transformation involves the application of specific mathematical techniques. This chapter is devoted to the mathematical fundamentals of map projection.

The coordinate systems of use for defining positions on the model of the Earth and the map are given. Then, the convention for defining azimuth is treated. The next topic is grid systems.

The basic unifying concepts for map projections are explored in terms of differential geometry. The process is sequential, going from arbitrary curves in space, to arbitrary surfaces in space, to general surfaces of revolution.

The differential geometry of curves is discussed to introduce needed concepts, such as radius of curvature and torsion of a space curve. The differential geometry of surfaces discussion introduces the first and second fundamental forms, as well as parametric curves and the condition of orthogonality. The surfaces of interest in mapping are surfaces of revolution. The general surface is particularized to surfaces of revolution. The process of transformation from nondevelopable to developable surfaces is considered. Representations of arc length, angles, and area, as well as the definition of the normal to the surface, are also discussed.

The basic transformation matrices are derived in forms useful for both the equal area and the conformal projections. The conditions of equality of area and conformality are mathematically defined. Then, a rotation method is discussed in which transverse and oblique projections may be obtained from the regular case. The convergency of the meridians in angular and linear form is then considered. The concept of the constant of the cone is introduced for the tangent and the secant case.

II. COORDINATE SYSTEMS AND AZIMUTH[7]

Coordinate systems are necessary to specify the location of positions on both the Earth and the map. In both cases, we are dealing with two-dimensional systems. Discussion of the two-dimensional characteristic for the model of the Earth is deferred until Chapter 3. In this section, the nomenclature and conventions required for the understanding of the following sections are introduced.

The terrestrial coordinate system is demonstrated in Figure 1. The origin,

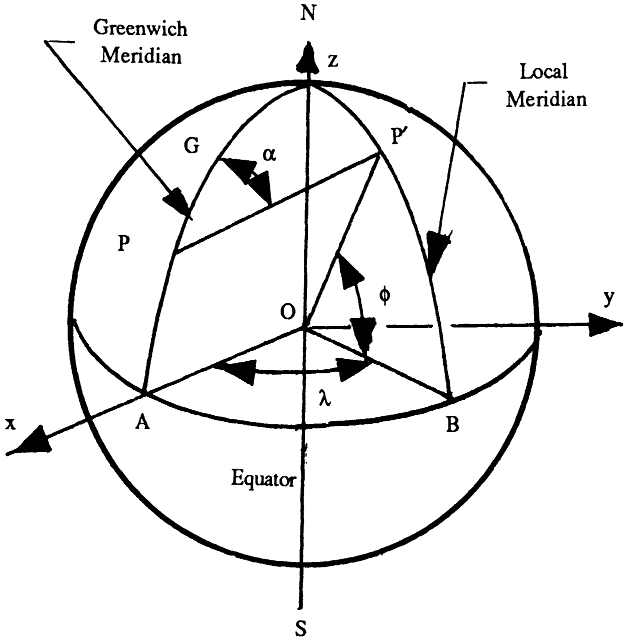

FIGURE 1. Terrestrial coordinate system.

O, of the system is at the center of the Earth. The x and y axes form the equatorial plane. The curve on the Earth formed by the intersection of this plane with the surface of the Earth is the equator. The positive x axis intersects the curve AGN. The curve AGN is a plane curve which is called the Greenwich meridian. The positive z axis coincides with the nominal axis of rotation of the Earth and points in the direction of the North Pole, N. The y axis completes a right-handed coordinate system.

A point P on the surface of the Earth is uniquely located by giving the latitude and longitude of the point. The latitude and longitude are independent angular coordinates.

A meridian is a curve formed by the intersection of a ficticious cutting plane containing the z axis and the surface of the Earth. The Greenwich meridian has already been mentioned. There is an infinity of meridians, depending on the orientation of the cutting plane. All meridians are perpendicular to the equatorial plane.

The use of latitude and longitude locates a point on a meridian and then locates the meridian with respect to the Greenwich meridian. Latitude is the angular measure defining the position of point P′ on the meridian BP′N. Latitude is denoted by ϕ. The position of the meridian that contains P is defined by the longitude, λ. The longitude is the angle AOB, measured in the equatorial plane, from the Greenwich meridian. The conventions for

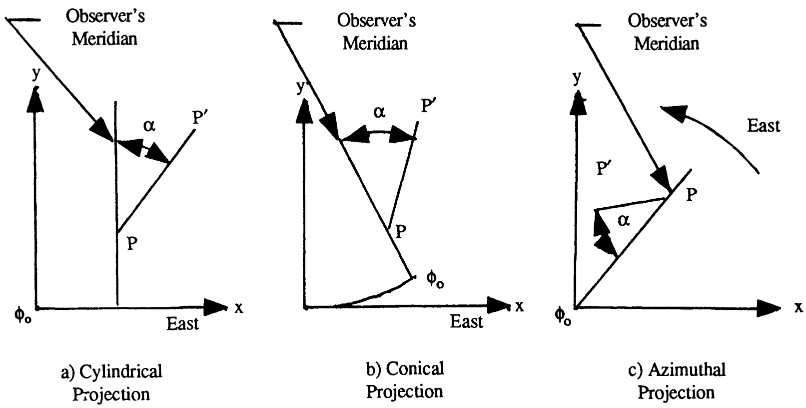

FIGURE 2. Mapping coordinate systems.

latitude and longitude are as follows: latitude is measured positive to the north, and negative to the south. Longitude is measured positive to the east, and negative to the west.

The circles of parallel are generated by cutting planes parallel to the equatorial plane which intersect the Earth. All points on a circle of parallel have the same latitude.

The coordinate system for the map (Figure 2) is a two-dimensional Cartesian system. The interpretation of the system differs for the projection schemes of Chapters 4, 5, and 6 that are portrayed.

For the cylindrical projections, and the world maps of Chapter 4, the line of the equator coincides with the x axis, where $\phi_o = 0°$. East is in the direction of positive x. The line of the central meridian defines the y axis, with north in the positive y direction. The central meridian, λ_o, is that reference meridian chosen by the cartographer to be at the center of the map. The intersection of the central meridian and equator defines the origin of the map.

Consider next a conical projection. In this case, the central meridian, λ_o, again coincides with the y axis, with y positive to the north. The origin is defined by the intersection of the central meridian with a circular arc representing a standard parallel of latitude, ϕ_o or ϕ, East is roughly in the plus x direction, as shown in Figure 2b.

Figure 2c indicates the convention for azimuthal projections. The y axis contains the central meridian. The origin coincides with the North Pole, or $\phi_o = 90°$. East is defined as a counterclockwise rotation.

The object of the practice of map projection is to transform from the terrestial angular coordinates, that is, the geographic coordinates, to a Cartesian map system in the direct transformation. The reverse is also introduced: transformation from Cartesian to geographic coordinates by the inverse transformation. Before this is done in Chapters 4, 5, and 6, more mathematical conventions and fundamentals must be introduced.

Azimuth is the means of measuring the direction of a second point from a given point and is an important quantity on both the model of the Earth and on the map. The azimuth, α, of point P′ in relation to point P is given in Figures 1 and 2 for both the Earth and the map. For the Earth, azimuth is the angle between the meridian containing point P, and the line between P and P′. In a like manner on the map, azimuth is the angle between the representation of the meridian containing P, and the line between P and P′. The convention used in this text is that azimuth is measured from the north in a clockwise manner. Azimuth ranges from 0° to 360°.

III. GRID SYSTEMS[1]

The parallels and meridians form a grid system on the model of the Earth. The grid serves as a reference system for locating points on the surface of the Earth.

In the map projection transformation process, this grid system is transferred to the map. Each projection scheme results in its own characteristic grid arrangement on the map. In Chapters 4, 5, and 6, after each set of plotting equations is derived, an empty grid system is given as an example for that particular projection. These grids give a clear indication of the geometric properties of each map projection.

On the map, the grids perform the same function as on the model of the Earth. They serve as a reference system for the location of points. In this text, the words "grid" and "graticule" are taken as equivalent in meaning.

IV. DIFFERENTIAL GEOMETRY OF SPACE CURVES[3]

It is first necessary to define a coordinate system at any point along a curve in space. The derivatives of the unit vectors are then required. With this information, we are able to define two important characteristics of the space curve: the torsion and the curvature.

The first step is to define a coordinate system located at any point along any space curve. Consider the space curve of Figure 3. Let ξ be an arbitrary parameter. Let the vector[7] to any point P on the curve in the Cartesian coordinate system be

$$\vec{r} = x(\xi)\hat{i} + y(\xi)\hat{j} + z(\xi)\hat{k} \tag{1}$$

Let $|\Delta \mathbf{r}| = \Delta s$. The unit tangent vector at point P is

$$\lim_{\Delta s \to 0} \frac{\Delta \vec{r}}{\Delta s} = \frac{d\vec{r}}{ds} = \hat{t} \tag{2}$$

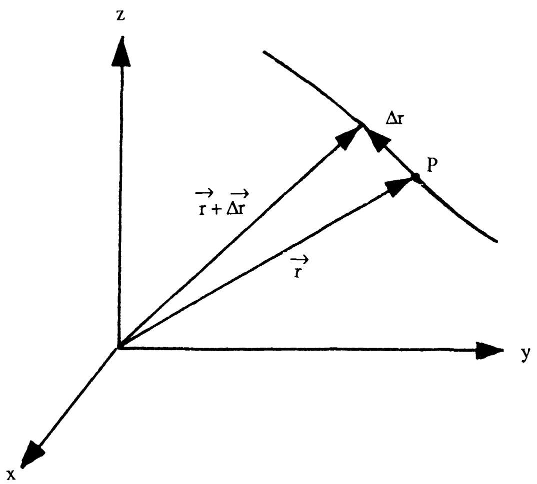

FIGURE 3. Geometry of a space curve.

Applying the chain rule to Equation 2, one finds

$$\hat{t} = \frac{d\vec{r}}{d\xi}\frac{d\xi}{ds} \tag{3}$$

Taking the total differential of Equation 1, we note

$$\hat{t} = \left(\frac{\partial x}{\partial \xi}\hat{i} + \frac{\partial y}{\partial \xi}\hat{j} + \frac{\partial z}{\partial \xi}\hat{k}\right)\left(\frac{d\xi}{ds}\right) \tag{4}$$

Upon taking the dot product of $\hat{t}$ with itself, we have

$$\hat{t}\cdot\hat{t} = 1 = \left[\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2\right]\left(\frac{d\xi}{ds}\right)^2$$

$$\left(\frac{d\xi}{ds}\right)^2 = \frac{1}{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}$$

$$\frac{d\xi}{ds} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}}$$

$$= \frac{1}{\left|\frac{\partial \vec{r}}{\partial \xi}\right|} \tag{5}$$

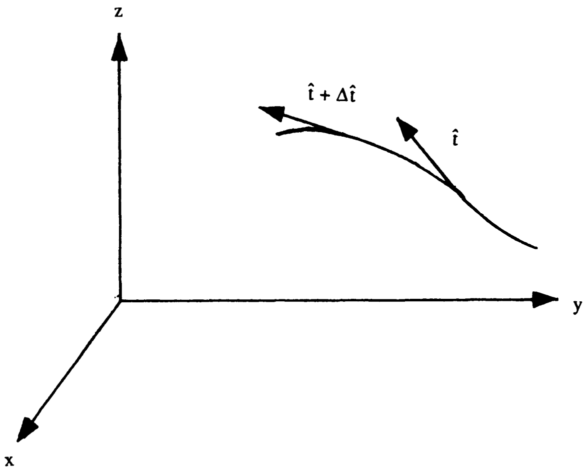

FIGURE 4. Consecutive unit tangents on a space curve.

Next, we look at two consecutive tangent vectors, as shown in Figure 4.

$$\lim_{\Delta s \to 0} \frac{\Delta \hat{t}}{\Delta s} = \frac{d\hat{t}}{ds}$$

Let

$$\frac{d\hat{t}}{ds} = -k\hat{n} \tag{6}$$

where k is defined as the curvature, and $\hat{n}$ is the principal unit normal. Note the relationship of the curvature and the normal to the curve.

Upon dotting $\hat{t}$ with itself, and differentiating, we find

$$\hat{t} \cdot \hat{t} = 1$$

$$2\hat{t} \cdot \frac{d\hat{t}}{ds} = 0$$

$$\hat{t} \cdot \hat{n} = 0$$

This means that $\hat{t}$ is perpendicular to $d\hat{t}/ds$, and from Equation 6, $\hat{n}$ is perpendicular to $\hat{t}$.

In order to obtain a right-handed triad, define the binormal vector.

$$\hat{b} = \hat{t} \times \hat{n} \tag{7}$$

Thus, we have the fundamental set of unit vectors $\hat{t}$, $\hat{n}$, and $\hat{b}$ associated with a continuous space curve at point P.

It is useful now to obtain the derivatives of the unit vectors as a function of distance along the curve. This is needed to obtain the torsion. From the definition of the unit vectors, we have the relations

$$\left.\begin{aligned} \hat{b} &= \hat{t} \times \hat{n} \\ \hat{t} &= \hat{n} \times \hat{b} \\ \hat{n} &= \hat{b} \times \hat{t} \end{aligned}\right\} \tag{8}$$

From the first of Equations 8,

$$\begin{aligned} \frac{d\hat{b}}{ds} &= \frac{d}{ds}(\hat{t} \times \hat{n}) \\ &= \frac{d\hat{t}}{ds} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \end{aligned} \tag{9}$$

Substitute Equation 6 into Equation 9 to obtain

$$\begin{aligned} \frac{d\hat{b}}{ds} &= -k\hat{n} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \\ &= \hat{t} \times \frac{d\hat{n}}{ds} \end{aligned} \tag{10}$$

Dot $\hat{n}$ with itself, and differentiate, so that

$$\hat{n} \cdot \hat{n} = 1$$

$$2\hat{n} \cdot \frac{d\hat{n}}{ds} = 0$$

Thus, $d\hat{n}/ds$ is perpendicular to $\hat{n}$ and must lie in the plane defined by $\hat{t}$ and $\hat{b}$ and has the components

$$\frac{d\hat{n}}{ds} = \psi\hat{t} + \tau\hat{b} \tag{11}$$

Substituting Equation 11 into Equation 10, we find

$$\frac{d\hat{b}}{ds} = \hat{t} \times (\psi\hat{t} + \tau\hat{b})$$

$$= \tau\hat{t} \times \hat{b}$$

$$= -\tau\hat{n} \qquad (12)$$

The variable τ is called the torsion. It is essentially a measure of the twist of the curve out of the plane defined by $\hat{n}$ and $\hat{t}$.

From the last of Equations 8,

$$\frac{d\hat{n}}{ds} = \frac{d}{ds}(\hat{b} \times \hat{t})$$

$$= \frac{d\hat{b}}{ds} \times \hat{t} + \hat{b} \times \frac{d\hat{t}}{ds} \qquad (13)$$

Substitute Equations 6 and 12 into Equation 13; this gives

$$\begin{aligned} \frac{d\hat{n}}{ds} &= -\tau\hat{n} \times \hat{t} + \hat{b} \times (-k\hat{n}) \\ &= \tau\hat{b} + k\hat{t} \end{aligned} \qquad (14)$$

Equations 6, 13, and 14 can be arranged in matrix form.

$$\begin{Bmatrix} d\hat{t}/ds \\ d\hat{n}/ds \\ d\hat{b}/ds \end{Bmatrix} = \begin{bmatrix} 0 & -k & 0 \\ k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{Bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{Bmatrix} \qquad (15)$$

These are the Frenet-Serret formulas.[7]

We now have the torsion and curvature as coefficients in vectorial relationships. The next step is to obtain the mathematical relations for the curvature and the torsion so they may be numerically evaluated.

The curvature, in general parametric form, is obtained from Equations 2 and 6.

$$-k\hat{n} = \frac{d\hat{t}}{ds}$$

$$= \frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right)$$

$$= \frac{d^2\vec{r}}{ds^2} \tag{16}$$

Taking the cross product of $\hat{t}$ with Equation 16, we have

$$\hat{t} \times (-k\hat{n}) = \hat{t} \times \left(\frac{d^2\vec{r}}{ds^2}\right) \tag{17}$$

Using the first of Equations 8 and Equation 2 in Equation 17,

$$-k\hat{b} = \frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}$$

$$k = \left|\left(\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right)\right| \tag{18}$$

For the general parametrization, we have the $\vec{r}$ derivatives

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{d\xi}\frac{d\xi}{ds} \tag{19}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d^2\vec{r}}{d\xi^2}\left(\frac{d\xi}{ds}\right)^2 + \frac{d\vec{r}}{d\xi}\left(\frac{d^2\xi}{ds^2}\right) \tag{20}$$

Note, substituting Equations 19 and 20 into Equation 18, that

$$k = \left|\frac{d\vec{r}}{d\xi}\frac{d\xi}{ds} \times \left[\frac{d^2\vec{r}}{d\xi^2}\left(\frac{d\xi}{ds}\right)^2 + \frac{d\vec{r}}{d\xi}\left(\frac{d^2\xi}{ds^2}\right)\right]\right|$$

$$= \left|\frac{d\vec{r}}{d\xi} \times \frac{d^2\vec{r}}{d\xi^2}\right|\left(\frac{d\xi}{ds}\right)^3 \tag{21}$$

Upon substituting Equation 5 into Equation 21, we arrive at

$$k = \frac{\left| \frac{d\vec{r}}{d\xi} \times \frac{d^2\vec{r}}{d\xi} \right|}{\left| \frac{d\vec{r}}{d\xi} \right|^3} \tag{22}$$

We now have a relationship that can be used to evaluate curvature. Of more use in map projections is the radius of curvature. The radius of curvature, R_c, is the reciprocal of the curvature.

$$R_c = \frac{1}{k} \tag{23}$$

Its utility is demonstrated in Chapter 3 with respect to curvature along a meridian, and perpendicular to a meridian.

A similar procedure can be followed to obtain the torsion. Upon taking the dot product of Equation 12 with n.

$$\tau = -\frac{d\hat{b}}{ds} \cdot \hat{n} \tag{24}$$

Substitute the first of Equation 8 into Equation 24; this gives

$$\begin{aligned} \tau &= -\frac{d}{ds}(\hat{t} \times \hat{n}) \cdot \hat{n} \\ &= -\left(\frac{d\hat{t}}{ds} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds}\right) \cdot \hat{n} \\ &= (\hat{t} \times \hat{n}) \cdot \frac{d\hat{n}}{ds} \end{aligned} \tag{25}$$

From Equation 16

$$\hat{n} = \frac{-\frac{d^2\vec{r}}{ds^2}}{k} \tag{26}$$

$$\frac{d\hat{n}}{ds} = \frac{-\frac{d^3\vec{r}}{ds^3}}{k} + \frac{\frac{dk}{ds}}{k^2}\frac{d^2\vec{r}}{ds^2} \tag{27}$$

Substitute Equations 2, 26, and 27 into Equation 25 to obtain

$$\tau = -\frac{d\vec{r}}{ds} \times \frac{\frac{d^2\vec{r}}{ds^2}}{k} \cdot \left(\frac{\frac{d^3\vec{r}}{ds^3}}{k} + \frac{\frac{dk}{ds}}{k^2} \frac{d^2\vec{r}}{ds^2} \right)$$

$$= \frac{1}{k^2} \left(\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} \right) \cdot \frac{d^3\vec{r}}{ds^3} \tag{28}$$

For the general parameterization, differentiate Equation 20

$$\frac{d^3\vec{r}}{ds^3} = \frac{d^3\vec{r}}{d\xi^3} \left(\frac{d\xi}{ds} \right)^3 + \frac{3d^2\vec{r}}{d\xi^2} \left(\frac{d\xi}{ds} \right) \left(\frac{d^2\xi}{ds^2} \right) + \frac{d\vec{r}}{d\xi} \frac{d^3\xi}{ds^3} \tag{29}$$

Substitute Equations 19, 20, and 29 into Equation 28 to obtain

$$\tau = \frac{\left[\left(\frac{d\vec{r}}{d\xi} \times \frac{d^2\vec{r}}{d\xi^2} \right) \cdot \left(\frac{d^3\vec{r}}{d\xi^3} \right) \right] \left(\frac{d\xi}{ds} \right)^6}{k^2} \tag{30}$$

Substituting Equation 5 into Equation 30 simplifies Equation 30 to

$$\tau = \frac{\left[\left(\frac{d\vec{r}}{d\xi} \times \frac{d^2\vec{r}}{d\xi} \right) \cdot \left(\frac{d^3\vec{r}}{d\xi^3} \right) \right]}{k^2 \left| \frac{d\vec{r}}{d\xi} \right|^6} \tag{31}$$

We now have a numerical relation for torsion. The torsion is important for such curves as the geodesic (Chapter 3). For plane curves, such as the meridian curve, and the equator, $\tau = 0$.

A. Example 1

As an example, consider a plane curve, and let $\xi = x$, and $y = y(x)$. The radius vector is

$$\mathbf{r} = x\hat{i} + y(x)\hat{j}$$

Find the radius of curvature and the torsion. Differentiate twice to obtain

$$\frac{d\vec{r}}{dx} = \hat{i} + \frac{dy}{dx}\hat{j}$$

$$\frac{d^2\vec{r}}{dx} = \frac{d^2y}{dx^2}\hat{j}$$

Substituting into Equation 22 to obtain

$$k = \frac{\left|\left(\hat{i} + \frac{dy}{dx}\hat{j}\right) \times \left(\frac{d^2y}{dx^2}\hat{j}\right)\right|}{\left[\left(\hat{i} + \frac{dy}{dx}\hat{j}\right) \cdot \left(\hat{i} + \frac{dy}{dx}\hat{j}\right)\right]^{3/2}}$$

$$= \frac{\left|\frac{d^2y}{dx^2}\hat{k}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

The radius of curvature is the reciprocal of the curvature. Thus,

$$R_c = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Taking the third derivative of the given equation,

$$\frac{d^3\vec{r}}{dx^3} = \frac{d^3y}{dx^3}\hat{j}$$

Substituting these relations into Equation 31,

$$\tau = \frac{\begin{vmatrix} 1 & \frac{dy}{dx} & 0 \\ 0 & \frac{d^2y}{dx^2} & 0 \\ 0 & \frac{d^3y}{dx^3} & 0 \end{vmatrix}}{k^2\left|\hat{i} + \frac{dx}{dy}\hat{j}\right|^6} = 0$$

This is no surprise, since we have a plane curve.

B. Example 2

Given: the plane curve $y = x^3 + 2x^2 + 3$.
Find: k, R_c, τ at $x = 3$.

$$\frac{dy}{dx} = (3x^2 + 4x)\big|_{x=3} = 39$$

$$\frac{d^2y}{dx^2} = (6x + 4)\big|_{x=3} = 22$$

$$k = \frac{\left|\dfrac{d^2y}{dx^2}\hat{k}\right|}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = \frac{22}{[1 + 39^2]^{3/2}}$$

$$= 0.0003705$$

$$R_c = \frac{1}{k} = 2{,}699$$

$$\tau = \frac{\begin{vmatrix} 1 & \dfrac{dy}{dx} & 0 \\ 0 & \dfrac{d^2y}{dx^2} & 0 \\ 0 & \dfrac{d^3y}{dx^3} & 0 \end{vmatrix}}{k^2\left|\dfrac{d\vec{r}}{dx}\right|^6} = 0$$

V. DIFFERENTIAL GEOMETRY OF A GENERAL SURFACE[3]

Parametric curves are defined for a general surface. Next, the tangents to these parametric curves are obtained. The tangent plane at a point follows from the tangents to the curves. Finally, a total differential is found as a function of the arbitrary parameters.

The parametric representation of a surface requires two parameters. Let the two arbitrary parameters for the parametric representation of the general surface be α_1 and α_2. Later, for the special surfaces of interest to mapping, α_1 and α_2 are identified with ϕ and λ, or latitude and longitude, respectively.

The vector to any point P on the surface is given by

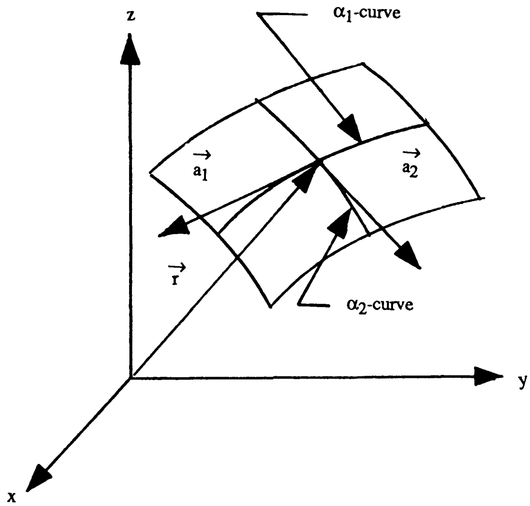

FIGURE 5. Parametric curves.

$$\vec{r} = \vec{r}(\alpha_1, \alpha_2) \tag{32}$$

If either of the two parameters is held constant, and the other one is varied, a space curve results. This space curve is the parametric curve. Figure 5 gives the parametric representation of space curves on a surface. The α_1 curve is the parametric curve along which α_2 is constant, and the α_2 curve is the parametric curve along which α_1 is constant.

The next step is to obtain the tangents to the parametric curves at point P. The tangent vector to the α_1 curve is

$$\vec{a}_1 = \frac{\partial \vec{r}}{\partial \alpha_1} \tag{33}$$

The tangent to the α_2 curve is

$$\vec{a}_2 = \frac{\partial \vec{r}}{\partial \alpha_2} \tag{34}$$

The plane spanned by the vectors $\vec{a}_1$ and $\vec{a}_2$ is the tangent plane to the surface at point P.

The total differential of Equation 32 is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \vec{r}}{\partial \alpha_2} d\alpha_2 \tag{35}$$

Substituting Equations 33 and 34 into Equation 35, we have

$$d\vec{r} = \vec{a}_1 d\alpha_1 + \vec{a}_2 d\alpha_2 \tag{36}$$

Armed with Equation 36, we are now ready to introduce the first fundamental form.

VI. FIRST FUNDAMENTAL FORM[3]

The first fundamental form of a surface is now to be derived. The first fundamental form is useful in dealing with arc length, area, angular measure on a surface, and the normal to the surface. Take the dot product of Equation 36 with itself to obtain a scalar.

$$\begin{aligned}(ds)^2 &= d\vec{r} \cdot d\vec{r} \\ &= (\vec{a}_1 d\alpha_1 + \vec{a}_2 d\alpha_2) \cdot (\vec{a}_1 d\alpha_1 + \vec{a}_2 d\alpha_2) \\ &= \vec{a}_1 \cdot \vec{a}_1 (d\alpha_1)^2 + \vec{a}_2 \cdot \vec{a}_2 (d\alpha_2)^2 + 2(\vec{a}_1 \cdot \vec{a}_2) d\alpha_1 d\alpha_2 \end{aligned} \tag{37}$$

Define new variables.

$$\left.\begin{aligned} E &= \vec{a}_1 \cdot \vec{a}_1 \\ F &= \vec{a}_1 \cdot \vec{a}_2 \\ G &= \vec{a}_2 \cdot \vec{a}_2 \end{aligned}\right\} \tag{38}$$

These new variables are known as the Gaussian fundamental quantities. These are defined for particular surfaces of interest to map projection. They are used in the derivation of the projections themselves and in the mathematical development of an estimate of distortion.

Substituting Equations 38 into Equation 37, we obtain the important differential form

$$(ds)^2 = E(d\alpha_1)^2 + 2F d\alpha_1 d\alpha_2 + G(d\alpha_2)^2 \tag{39}$$

Equation 39 is the first fundamental form of a surface, and this is very useful through the whole process of map projection. The first fundamental form is now applied to linear measure on any surface.

Arc length can be found immediately from the integration of Equation 39. The distance between two arbitrary points P_1 and P_2 on the surface is given by

$$s = \int_{P_1}^{P_2} \sqrt{E(d\alpha_1)^2 + 2F d\alpha_1 d\alpha_2 + G(d\alpha_2)^2}$$

$$= \int_{P_1}^{P_2} \left\{ \sqrt{E + 2F\left(\frac{d\alpha_2}{d\alpha_1}\right) + G\left(\frac{d\alpha_1}{d\alpha_2}\right)^2} \right\} d\alpha_1 \qquad (40)$$

Equation 40 is useful as soon as $d\alpha_2/d\alpha_1$ is defined, and it is used in Chapter 3 for distance along the spheroid.

Angles between two unit tangents $\hat{a}_1$ and $\hat{a}_2$ on the surface can be found by taking the dot product of Equations 33 and 34 and applying Equation 38.

$$\cos\theta = \frac{\vec{a}_1}{|\vec{a}_1|} \cdot \frac{\vec{a}_2}{|\vec{a}_2|} \qquad (41)$$

$$= \frac{F}{\sqrt{EG}} \qquad (42)$$

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

$$= \sqrt{1 - \frac{F^2}{EG}}$$

$$= \sqrt{\frac{EG - F^2}{EG}} \qquad (43)$$

Define

$$H = EG - F^2 \qquad (44)$$

and substitute Equation 44 into Equation 43 to obtain

$$\sin\theta = \sqrt{\frac{H}{EG}} \qquad (45)$$

The normal to the surface at point P is

$$\hat{n} = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|}$$

$$= \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1||\vec{a}_2|\sin\theta} \qquad (46)$$

Substituting Equations 38 and 45 into Equation 46, we have

$$\hat{n} = \frac{\vec{a}_1 \times \vec{a}_2}{\sqrt{E}\,\sqrt{G}\sqrt{\dfrac{H}{EG}}}$$

$$= \frac{\vec{a}_1 \times \vec{a}_2}{\sqrt{H}} \tag{47}$$

Incremental area can be obtained by a consideration of incremental distance along the parametric curves. Along the α_2 curve and the α_1 curve, respectively,

$$ds_1 = \sqrt{E}\, d\alpha_1$$

$$ds_2 = \sqrt{G}\, d\alpha_2 \tag{48}$$

The differential area is then

$$dA = ds_1 ds_2 \sin\theta$$

$$= \sqrt{EG}\, \sin\theta d\alpha_1 d\alpha_2 \tag{49}$$

Substituting Equation 45 into Equation 49, we have

$$dA = \sqrt{EG}\sqrt{\frac{H}{EG}}\, d\alpha_1 d\alpha_2$$

$$= \sqrt{H}\, d\alpha_1 d\alpha_2 \tag{50}$$

Thus, the first fundamental form has given a means to derive the arc length, the unit normal to the surface at every point, and incremental area. In conjunction with the second fundamental form of the next section, it is useful in determining the radii of curvature of the surface.

Assume now that $\vec{a}_1$, and $\vec{a}_2$ are chosen to be orthogonal. Thus $\theta = 90°$. If this is substituted in Equation 42

$$\cos 90° = \frac{F}{\sqrt{EG}} = 0$$

$$F = 0$$

When orthogonality is present, Equation 50 is simplified. The first fundamental form is then

$$(ds)^2 = E(d\alpha_1)^2 + G(d\alpha_2)^2 \tag{51}$$

This is the case for the practical, parametric systems of map projection. Thus, the form of Equation 51 always applies.

A. Example 3

The first fundamental form for a planar surface in Cartesian coordinates or a cylindrical surface is

$$(ds)^2 = (dx)^2 + (dy)^2$$

Thus, the fundamental quantities are

$$E = 1$$

$$G = 1$$

B. Example 4

The first fundamental form for a plane surface in polar coordinates or a conical surface is

$$(ds)^2 = (d\rho)^2 + \rho^2(d\theta)^2$$

The first fundamental quantities are

$$E = 1$$

$$G = \rho^2$$

C. Example 5

The first fundamental form for a sphere in geographic coordinates is

$$(ds)^2 = R^2(d\phi)^2 + R^2\cos^2\theta(d\lambda)^2$$

Thus, the fundamental quantities are

$$E = R^2$$

$$G = R^2\cos^2\phi$$

D. Example 6

The first fundamental form for a spheroid in geographic coordinates is

$$(ds)^2 = R_m^2(d\phi)^2 + R_p^2\cos^2\phi(d\lambda)^2$$

Thus, the fundamental quantities are

$$E = R_m^2$$

$$G = R_p^2 \cos^2\phi$$

E. Example 7

Given: the first fundamental form for the sphere.
Find: distance between P_1 and P_2 and incremental area.

$$s = \int_{P_1}^{P_2} \sqrt{R^2(d\phi)^2 + R^2\cos^2\phi(d\lambda)^2}$$

$$= R\int_{P_1}^{P_2} \sqrt{1 + \cos^2\phi\left(\frac{d\lambda}{d\phi}\right)^2}\, d\phi$$

$$dA = \sqrt{R^2R^2\cos^2\phi}\, d\phi d\lambda$$

$$= R^2\cos\phi d\phi d\lambda$$

VII. SECOND FUNDAMENTAL FORM

The second fundamental form provides the means for evaluating the curvature of a surface and determining the principal directions on the surface. The section begins by defining the second fundamental forms for a general surface. The second fundamental form, in conjunction with the first fundamental form, defines the curvature. The second fundamental quantities are then defined. The principal directions on the surface are then obtained by an optimization process. Since the two principal directions are found to be orthogonal, the curvatures in the two principal directions are given by simplified equations.

We begin the derivation for the curvature of a surface with a consideration of normal sections through the surface. A normal section implies that the normal to the parametric curve and the surface coincide. We start with the normal at an arbitrary point. For the parametric curve, take the dot product of $\hat{n}$ with Equation 6.

$$\hat{n} \cdot \frac{d\hat{t}}{ds} = -k\hat{n} \cdot \hat{n}$$

$$= -k \qquad (51a)$$

Substitute the derivative of Equation 2 into Equation 51 to obtain

$$k = -\frac{d^2\vec{r}}{ds^2} \cdot \hat{n} \qquad (52)$$